Symmetries in Nuclei

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École Joliot Curie, September 2010

Symmetries in Nuclei

(Dynamical) symmetries in quantum mechanics

Symmetries of the nuclear shell model

(Symmetries of the interacting boson model)

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(Dynamical) symmetries in quantum mechanics

Symmetry in quantum mechanics

The harmonic oscillator

Isospin symmetry in nuclei

Dynamical symmetry

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Symmetry in quantum mechanics

Assume a hamiltonian *H* which commutes with operators g_i that form a Lie algebra *G*:

$$\forall \hat{g}_i \in G: \quad \left[\hat{H}, \hat{g}_i\right] = 0$$

$\therefore$ *H* has *symmetry G* or is *invariant under G*.

Lie algebra: a set of (infinitesimal) operators that closes under commutation.

Consequences of symmetry

Degeneracy structure: If $|\gamma\rangle$ is an eigenstate of *H* with energy *E*, so is $g_i|\gamma\rangle$:

$$\hat{H}|\gamma\rangle = E|\gamma\rangle \Rightarrow \hat{H}\hat{g}_i|\gamma\rangle = \hat{g}_i\hat{H}|\gamma\rangle = E\hat{g}_i|\gamma\rangle$$

Degeneracy structure and labels of eigenstates of *H* are determined by algebra *G*:

$$\hat{H}|\Gamma\gamma\rangle = E(\Gamma)|\Gamma\gamma\rangle; \quad \hat{g}_i|\Gamma\gamma\rangle = \sum_{\gamma'} a^{\Gamma}_{\gamma'\gamma}(i)|\Gamma\gamma'\rangle$$

Casimir operators of *G* commute with all g_i:

$$\hat{H} = \sum_m \mu_m \hat{C}_m[G]$$

The (3D) harmonic oscillator

The hamiltonian of the harmonic oscillator is

$$\hat{H} = \frac{p^2}{2M} + \frac{1}{2} M\omega^2 r^2$$

Standard wave quantum mechanics gives

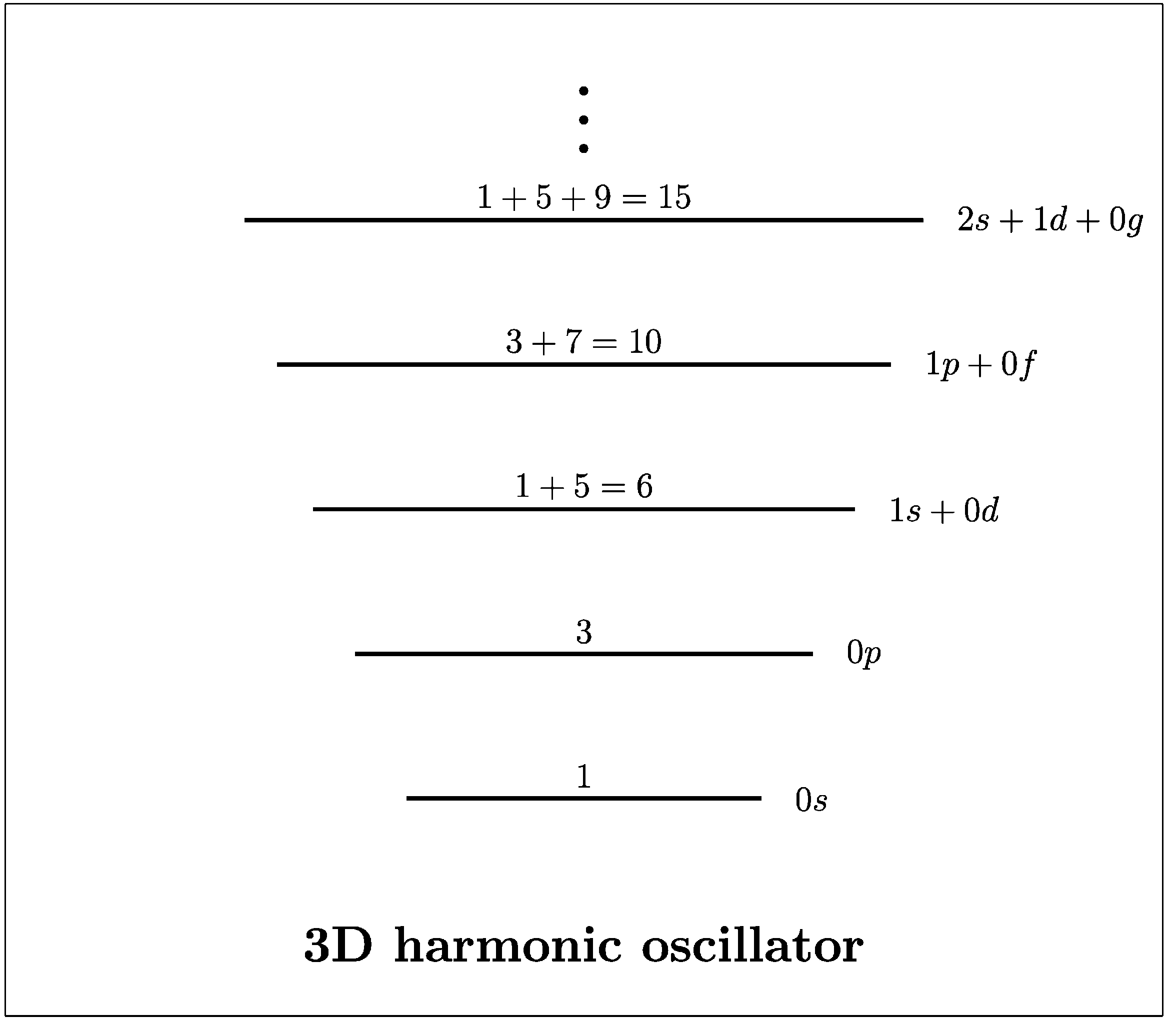
$$\hat{H}\Psi_{nlm}(r,\theta,\varphi) = \left(2n + l + \frac{3}{2}\right)\hbar\omega\Psi_{nlm}(r,\theta,\varphi)$$

$$\text{with } n = 0,1,\ldots\ ;\ l = 0,1,\ldots\ ;\ m = -l,\ldots,+l$$

Degeneracy in *m* originates from rotational symmetry. Additional degeneracy for all (*n*,*l*) combinations with *2n*+*l*=*N*.

What is the origin of this degeneracy?

Degeneracies of the 3D HO



Raising and lowering operators

Introduce the raising and lowering operators

$$b_x^+ = \frac{1}{\sqrt{2}}\left(x' - \frac{\partial}{\partial x'}\right), \quad b_y^+ = \frac{1}{\sqrt{2}}\left(y' - \frac{\partial}{\partial y'}\right), \quad b_z^+ = \frac{1}{\sqrt{2}}\left(z' - \frac{\partial}{\partial z'}\right)$$

$$b_x = \frac{1}{\sqrt{2}}\left(x' + \frac{\partial}{\partial x'}\right), \quad b_y = \frac{1}{\sqrt{2}}\left(y' + \frac{\partial}{\partial y'}\right), \quad b_z = \frac{1}{\sqrt{2}}\left(z' + \frac{\partial}{\partial z'}\right)$$

with $\quad x' = x/l, y' = y/l, z' = z/l; \quad l = \sqrt{\frac{\hbar}{M\omega}}$

The 3D HO hamiltonian becomes

$$\hat{H} = \frac{p^2}{2M} + \frac{1}{2}M\omega^2 r^2 = \sum_{i=x,y,z}\left(b_i^+ b_i + \frac{1}{2}\right)\hbar\omega$$

M. Moshinsky, *The Harmonic Oscillator in Modern Physics*

U(3) symmetry of the 3D HO

The raising and lowering operators satisfy

$$\left[b_i, b_j\right] = 0, \quad \left[b_i^+, b_j^+\right] = 0, \quad \left[b_i, b_j^+\right] = \delta_{ij}$$

The bilinear combinations u_{ij} commute with H:

$$\hat{u}_{ij} \equiv b_i^+ b_j \Rightarrow \left[\hat{u}_{ij}, \hat{H}\right] = 0, \quad \forall i, j \in \{x, y, z\}$$

The nine operators u_{ij} generate the algebra U(3):

$$\left[\hat{u}_{ij}, \hat{u}_{kl}\right] = \hat{u}_{il}\delta_{jk} - \hat{u}_{kj}\delta_{il}$$

$\Rightarrow$ The symmetry of the harmonic oscillator in 3 dimensions is U(3).

The U(3)=U(1)⊕SU(3) algebra

The generators u_{ij} of U(3) can be written as

$$b_x^+ b_x + b_y^+ b_y + b_z^+ b_z = \frac{\hat{H}}{\hbar\omega} - \frac{3}{2}$$

$$\hat{L}_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = -i\hbar\left(b_x^+ b_y - b_y^+ b_x\right) \quad + \text{cyclic}$$

$$\hat{Q}_0 = \hbar\left(2b_z^+ b_z - b_x^+ b_x - b_y^+ b_y\right)$$

$$\hat{Q}_{\mp 1} = \hbar\sqrt{\frac{3}{2}}\left(\pm b_z^+ b_x \pm b_x^+ b_z - i b_y^+ b_z - i b_z^+ b_y\right)$$

$$\hat{Q}_{\mp 2} = \hbar\sqrt{\frac{3}{2}}\left(b_x^+ b_x - b_y^+ b_y \mp i b_x^+ b_y \mp i b_y^+ b_x\right)$$

Many particles in the 3D HO

Define operators for each particle k=1,2,…,A:

$$b_{x,k}^{+} = \frac{1}{\sqrt{2}}\left(x_k' - \frac{\partial}{\partial x_k'}\right), \quad b_{y,k}^{+} = \frac{1}{\sqrt{2}}\left(y_k' - \frac{\partial}{\partial y_k'}\right), \quad b_{z,k}^{+} = \frac{1}{\sqrt{2}}\left(z_k' - \frac{\partial}{\partial z_k'}\right)$$

$$b_{x,k} = \frac{1}{\sqrt{2}}\left(x_k' + \frac{\partial}{\partial x_k'}\right), \quad b_{y,k} = \frac{1}{\sqrt{2}}\left(y_k' + \frac{\partial}{\partial y_k'}\right), \quad b_{z,k} = \frac{1}{\sqrt{2}}\left(z_k' + \frac{\partial}{\partial z_k'}\right)$$

The *total* U(3) algebra is generated by

$$\sum_{k=1}^{A} b_{i,k}^{+} b_{j,k}, \quad i,j \in \{x,y,z\}$$

Many particles in the 3D HO

Many-body hamiltonian with U(3) symmetry:

$$\hat{H} = \hbar\omega\left(\sum_{k=1}^{A} b^{+}_{x,k} b_{x,k} + b^{+}_{y,k} b_{y,k} + b^{+}_{z,k} b_{z,k}\right) + \sum_{k<l=1}^{A} \hat{V}(k,l)$$

$$\left[\hat{H}, \sum_{k=1}^{A} b^{+}_{i,k} b_{j,k}\right], \quad \forall i,j \in \{x,y,z\}$$

This property is valid if the interaction equals

$$\hat{C}_2[\mathrm{SU}(3)] = \frac{1}{2}\boldsymbol{L}\cdot\boldsymbol{L} + \frac{1}{6}\boldsymbol{Q}\cdot\boldsymbol{Q} = \sum_{k,l=1}^{A}\left(\frac{1}{2}\boldsymbol{L}(k)\cdot\boldsymbol{L}(l) + \frac{1}{6}\boldsymbol{Q}(k)\cdot\boldsymbol{Q}(l)\right)$$

J.P. Elliott, Proc. Roy. Soc. A **245** (1958) 128; 562

Dynamical symmetry

Two algebras $G_1 \supset G_2$ and a hamiltonian

$$\hat{H} = \sum_m \mu_m \hat{C}_m [G_1] + \sum_n \nu_n \hat{C}_n [G_2]$$

$\therefore$ *H* has symmetry G_2 but *not* G_1!

Eigenstates are independent of parameters μ_m and ν_n in *H*.

Dynamical symmetry breaking *"splits but does not admix eigenstates"*.

Isospin symmetry in nuclei

Empirical observations:

About equal masses of n(eutron) and p(roton).

n and p have spin 1/2.

Equal (to about 1%) nn, np, pp strong forces.

This suggests an *isospin* SU(2) symmetry of the nuclear hamiltonian:

$$\mathrm{n}: \quad t = \tfrac{1}{2}, m_t = +\tfrac{1}{2}; \qquad \mathrm{p}: \quad t = \tfrac{1}{2}, m_t = -\tfrac{1}{2}$$

$$\Rightarrow \quad \hat{t}_+\mathrm{n} = 0, \quad \hat{t}_+\mathrm{p} = \mathrm{n}, \quad \hat{t}_-\mathrm{n} = \mathrm{p}, \quad \hat{t}_-\mathrm{p} = 0, \quad \hat{t}_z n = \tfrac{1}{2}\mathrm{n}, \quad \hat{t}_z p = -\tfrac{1}{2}\mathrm{p}$$

W. Heisenberg, Z. Phys. **77** (1932) 1
E.P. Wigner, Phys. Rev. **51** (1937) 106

Isospin SU(2) symmetry

Isospin operators form an SU(2) algebra:

$$\left[\hat{t}_z,\hat{t}_\pm\right]=\pm\hat{t}_\pm,\quad \left[\hat{t}_+,\hat{t}_-\right]=2\hat{t}_z$$

Assume the nuclear hamiltonian satisfies

$$\left[\hat{H}_{\mathrm{nucl}},\hat{T}_\nu\right]=0,\quad \hat{T}_\nu=\sum_{k=1}^{A}\hat{t}_\nu(k)$$

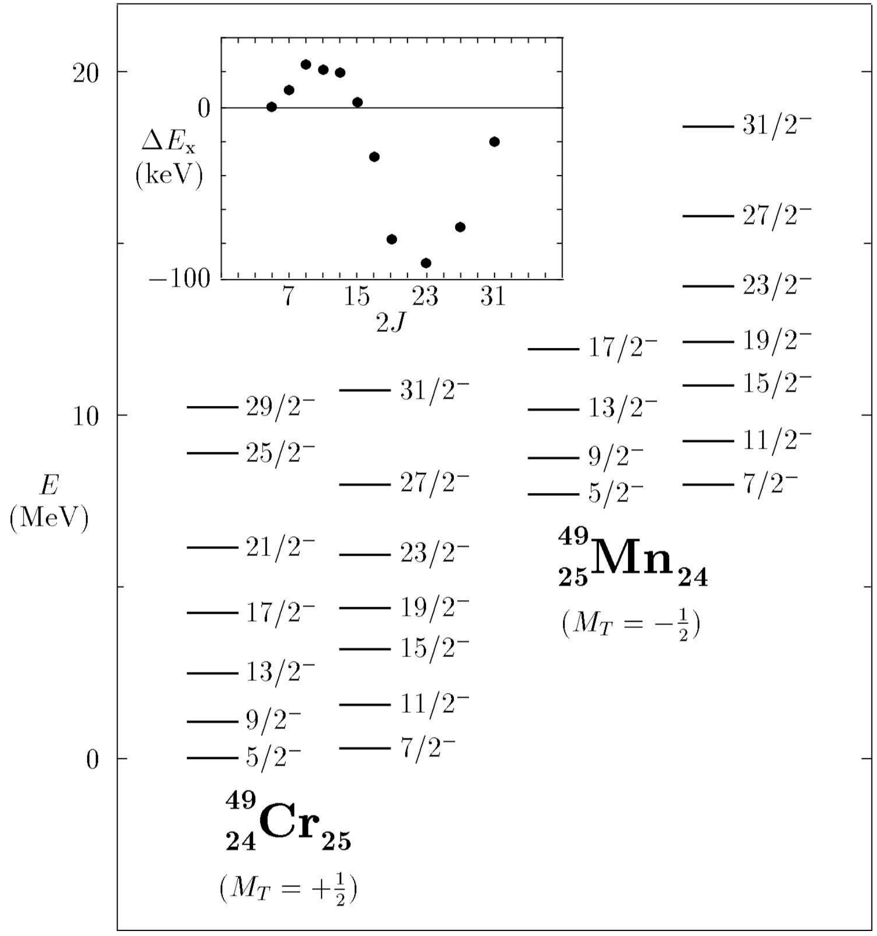
∴ H_{nucl} has SU(2) symmetry with degenerate states belonging to isobaric multiplets:

$$\left|\eta T M_T\right\rangle,\quad M_T=-T,-T+1,\ldots,+T$$

Isospin symmetry breaking: *A*=49

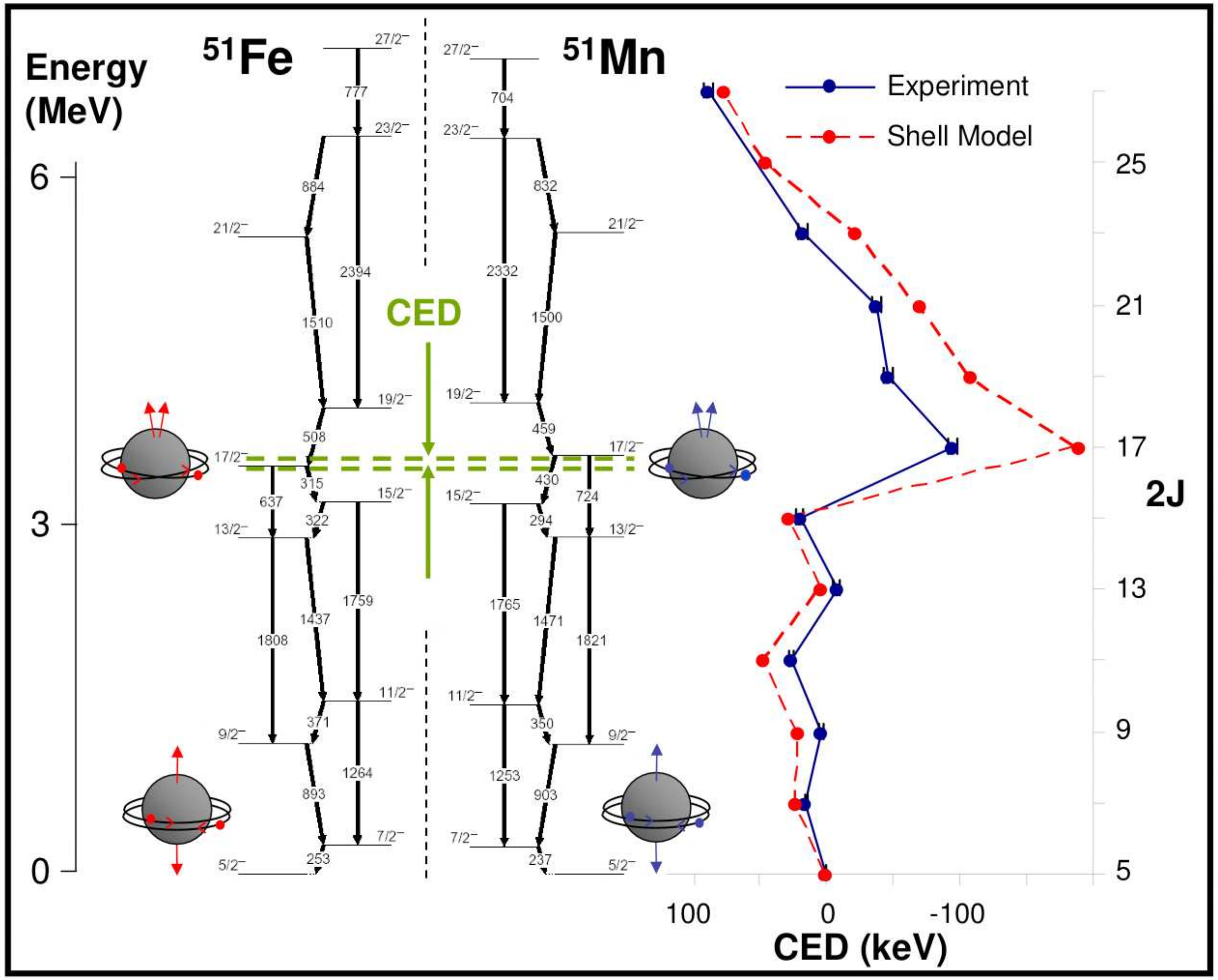
Empirical evidence from isobaric multiplets.

Example: *T*=1/2 doublet of *A*=49 nuclei.



O'Leary *et al.*, Phys. Rev. Lett. **79** (1997) 4349

Isospin symmetry breaking: *A*=51



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Isospin SU(2) dynamical symmetry

Coulomb interaction can be approximated as

$$\hat{H}_{\mathrm{Coul}} \approx \kappa_0 + \kappa_1 \hat{T}_z + \kappa_2 \hat{T}_z^2 \Rightarrow \left[\hat{H}_{\mathrm{Coul}}, \hat{T}_z\right] = 0, \quad \left[\hat{H}_{\mathrm{Coul}}, \hat{T}_\pm\right] \neq 0$$

$\therefore$ H_{nucl}+H_{Coul} has SU(2) dynamical symmetry and SO(2) symmetry.

M_T-degeneracy is lifted according to

$$\hat{H}_{\mathrm{Coul}} \left| \eta T M_T \right\rangle = \left(\kappa_0 + \kappa_1 M_T + \kappa_2 M_T^2\right) \left| \eta T M_T \right\rangle$$

Summary of labelling:

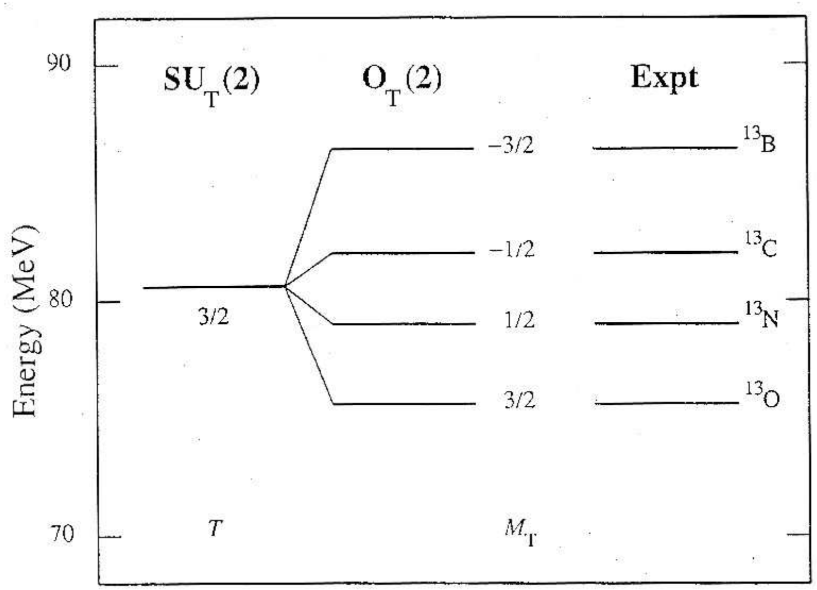
$$\begin{array}{ccc} \mathrm{SU}(2) & \supset & \mathrm{SO}(2) \\ \downarrow & & \downarrow \\ T & & M_T \end{array}$$

Isobaric multiplet mass equation

Isobaric multiplet mass equation:

$$E(\eta T M_T) = \kappa(\eta, T) + \kappa_1 M_T + \kappa_2 M_T^2$$

Example: *T*=3/2 multiplet for *A*=13 nuclei.



E.P. Wigner, Proc. Welch Found. Conf. (1958) 88

Isospin selection rules

Internal E1 transition operator is isovector:

$$\hat{T}_\mu^{\mathrm{E1}} = \sum_{k=1}^{A} e_k \, r_\mu(k) = \frac{e}{2}\left(\underbrace{\sum_{k=1}^{A} r_\mu(k)}_{\text{CM motion}} + \underbrace{2\sum_{k=1}^{A} \hat{t}_z(k) r_\mu(k)}_{\text{isovector}} \right)$$

Selection rule for *N=Z* (*M*$_T$=0) nuclei: *No E1 transitions are allowed between states with the same isospin.*

L.E.H. Trainor, Phys. Rev. **85** (1952) 962
L.A. Radicati, Phys. Rev. **87** (1952) 521

E1 transitions and isospin mixing

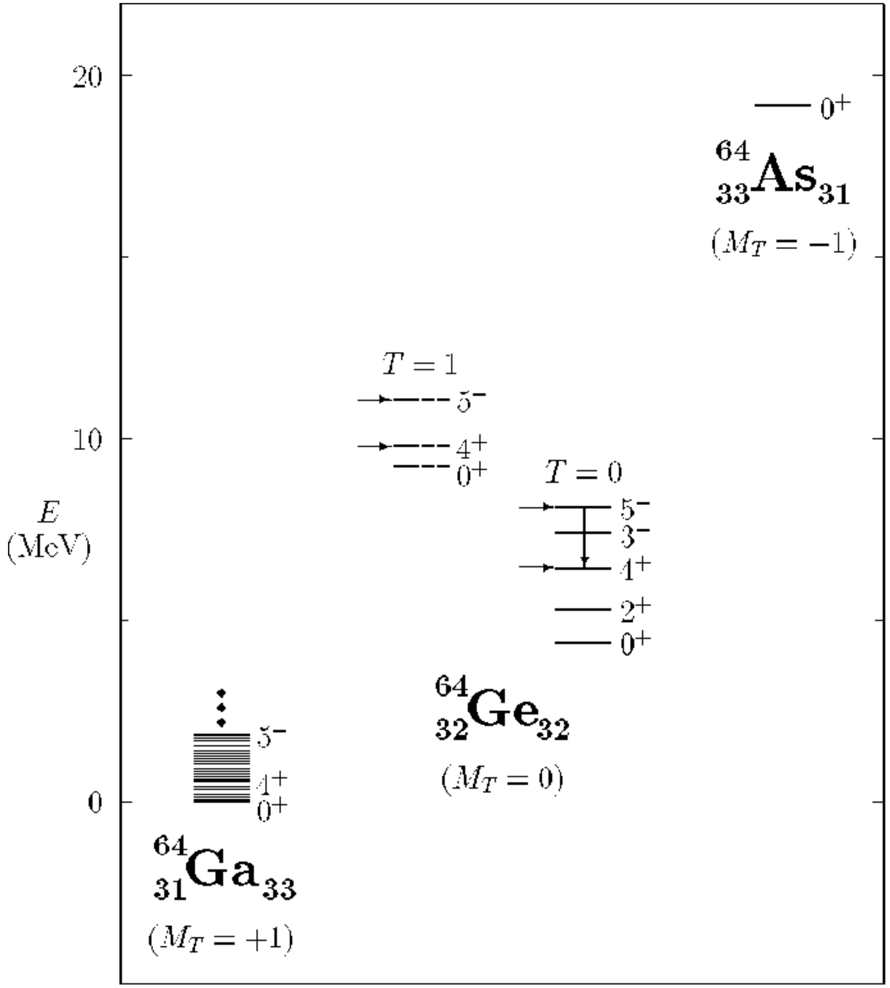
B(E1;$5^- \rightarrow 4^+$) in ^{64}Ge from:

- *lifetime of 5^- level;*
- *δ(E1/M2) mixing ratio of $5^- \rightarrow 4^+$ transition;*
- *relative intensities of transitions from 5^-.*

Estimate of minimum isospin mixing:

$$P\left(T=1,5^-\right) \approx P\left(T=1,4^+\right)$$

$$\approx 2.5\%$$



E.Farnea *et al.*, Phys. Lett. B **551** (2003) 56

Dynamical algebra

Take a generic many-body hamiltonian:

$$\hat{H} = \sum_i \varepsilon_i c_i^+ c_i + \frac{1}{4}\sum_{ijkl} \upsilon_{ijkl} c_i^+ c_j^+ c_l c_k + \cdots$$

Rewrite *H* as (bosons: *q*=0; fermions: *q*=1)

$$\hat{H} = \sum_{il}\left(\varepsilon_i \delta_{il} - (-)^q \frac{1}{4}\sum_j \upsilon_{ijlk}\right)\hat{u}_{il} + (-)^q \frac{1}{4}\sum_{ijkl} \upsilon_{ijlk} \hat{u}_{ik} \hat{u}_{jl} + \cdots$$

Operators u_{ij} generate the *dynamical algebra* U(*n*) for bosons and for fermions (*q*=0,1):

$$\hat{u}_{ij} \equiv c_i^+ c_j \Rightarrow \left[\hat{u}_{ij}, \hat{u}_{kl}\right] = \hat{u}_{il}\delta_{jk} - \hat{u}_{kj}\delta_{il}$$

Dynamical symmetry (DS)

With each chain of *nested* algebras

$$\mathrm{U}(n) = G_{\mathrm{dyn}} = G_1 \supset G_2 \supset \cdots \supset G_{\mathrm{sym}}$$

…is associated a *particular* class of many-body hamiltonian

$$\hat{H} = \sum_m \mu_m \hat{C}_m[G_1] + \sum_n \nu_n \hat{C}_n[G_2] + \cdots$$

Since *H* is a sum of commuting operators

$$\forall m,n,a,b: \quad \left[\hat{C}_m[G_a], \hat{C}_n[G_b]\right] = 0$$

…it can be solved analytically!

DS in nuclear physics

Name	G_{dyn}	G_{break}	G_{sym}	Application	Reference
Isospin	SU(2)	—	SO(2)	Isobaric multiplets, IMME	Heisenberg [4] Wigner [5]
Quasi-spin	SU(2)	—	SO(2)	Seniority spectra	Racah [6] Kerman [7]
supermultiplet SU(3) model	U(4Ω)	SU(3)	SO(3)	Wigner energy Rotational bands	Wigner [8] Elliott [9]
Interacting Boson Model	U(6)	U(5) SU(3) SO(6)	SO(3)	Vibrational nuclei Rotational nuclei γ-unstable nuclei	Arima and Iachello [10]
F-spin	SU(2)	—	SO(2)	F-spin multiplets, FMME	Brentano *et al.* [11]