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Lacanau, 28.09.2009

Modern Theory of Nuclear Forces

Lecture 1: Introduction & first look into ChPT

- Historical overview
- Conventional approach
  - General structure of the 2N force
  - Modern “high-precision“ NN potentials
  - Beyond two nucleons
- Chiral Perturbation Theory
  - Introduction
  - Chiral symmetry of QCD
  - Effective Lagrangian

Lecture 2: EFTs for two nucleons

Lecture 3: Nuclear forces from chiral EFT

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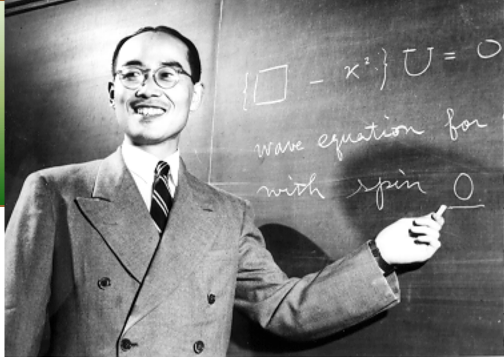
Historical overview



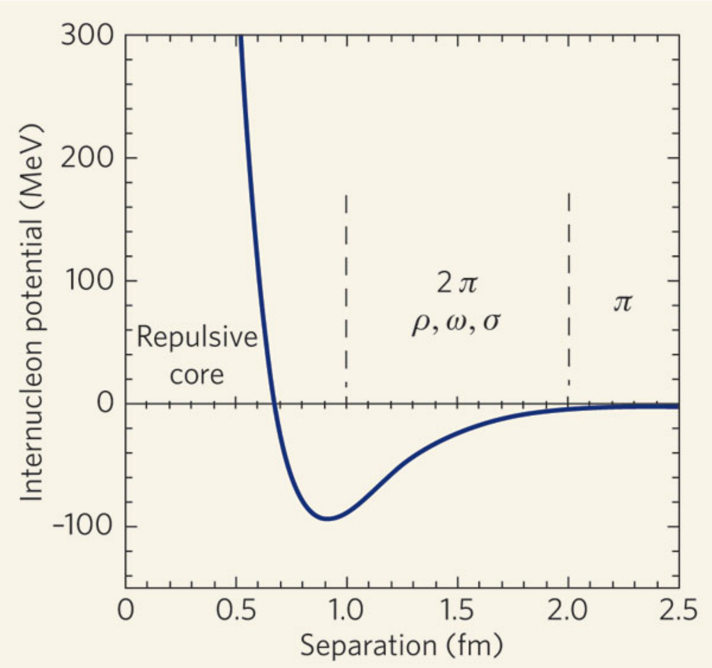
1935 *Yukawa* suggests that nucleons interact via exchange of massive scalar particles

1936-42 Extension to pseudoscalar/pseudovector exchange particles by *Proca, Kemmer, Moller, Rosenfeld and Schwinger*

1946 Existence of an isovector pseudoscalar meson (pion) predicted by *Pauli*

1947 Experimental discovery of pions by *Lattes, Muirhead, Occhialini and Powell*

1951 *Taketani, Nakamura, Sasaki* introduce new concept: long-range (1π), medium-range $(1\pi + 2\pi)$ and core (???)



1950s 2π-exchange potential studied by *Taketani et al., Brückner, Watson, ...*

1960s Discovery of heavy mesons, OBE models

70s, 80s Dispersion and inverse scattering theory, BE and quark cluster models, phenomenology

80s, 90s High-precision potentials $(\chi^2_{\text{data}} \sim 1)$: AV18, CD Bonn, Nijm I,II, Reid 93, …

since 91 Chiral effective field theory

2N force: general structure

Available vectors: $\vec r_1,\ \vec r_2,\ \vec p_1,\ \vec p_2,\ \vec\sigma_1,\ \vec\sigma_2$ and isovectors: $\boldsymbol\tau_1\,,\ \boldsymbol\tau_2$

Invariance under translations and Galilei transformations: $V_{2N}(\vec r_1,\ \vec r_2,\ \vec p_1,\ \vec p_2) = V_{2N}(\vec r,\ \vec p)$

where $\vec r = \vec r_1 - \vec r_2\,, \quad \vec p = \frac{1}{2}(\vec p_1 - \vec p_2\,) = -i\vec\nabla_r$

Invariance under rotations, space reflection, time reversal & isospin rotations

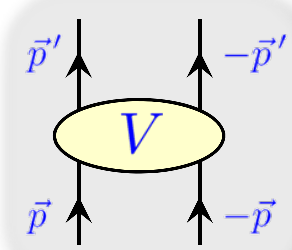
$$\Longrightarrow \quad \Big\{\underbrace{1,\quad \vec\sigma_1\cdot\vec\sigma_2\,,\quad S_{12}(\vec r\,)\,,\quad S_{12}(\vec p\,)\,,\quad \vec L\cdot\vec S\,,\quad (\vec L\cdot\vec S)^2}_{\textit{spin-space}}\Big\} \otimes \Big\{\underbrace{1,\quad \boldsymbol\tau_1\cdot\boldsymbol\tau_2}_{\textit{isospin}}\Big\}$$

where: $\vec L = \vec r\times\vec p\,, \quad \vec S = 1/2(\vec\sigma_1+\vec\sigma_2)\,, \quad S_{12}(\vec x\,) = 3(\vec\sigma_1\cdot\hat x)(\vec\sigma_2\cdot\hat x) - \vec\sigma_1\cdot\vec\sigma_2$

All operators are to be multiplied with scalar functions of $r^2,\ p^2,\ \vec r\cdot\vec p$ or, equivalently, $r^2,\ p^2,\ L^2$ since $(\vec r\cdot\vec p\,)^2 = r^2p^2 - L^2$ such that the resulting V is hermitian.

Momentum-space representation $\langle\vec p\,'|V|\vec p\,\rangle$

$$\Big\{\underbrace{1,\quad \vec\sigma_1\cdot\vec\sigma_2\,,\quad S_{12}(\vec q\,)\,,\quad S_{12}(\vec k\,)\,,\quad i\vec S\cdot\vec q\times\vec k\,,\quad \vec\sigma_1\cdot\vec q\times\vec k\,\vec\sigma_2\cdot\vec q\times\vec k}_{\textit{spin-momentum}}\Big\} \otimes \Big\{\underbrace{1,\quad \boldsymbol\tau_1\cdot\boldsymbol\tau_2}_{\textit{isospin}}\Big\}$$



where $\vec q = \vec p\,' - \vec p\,, \quad \vec k = \vec p\,' + \vec p\,$.

The operators are to be multiplied with scalar functions of $\vec q^{\,2},\ \vec k^{\,2},\ \vec q\cdot\vec k$.

2N force: isospin structure

Henley, Miller '79

Class I (isospin invariant forces): $[V_I^{\rm 2N},\, \boldsymbol{T}] = 0 \Longrightarrow$ $\boxed{V_I^{\rm 2N} = \alpha + \beta(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2)}$

Class II (charge independence breaking):

$$[V_{II}^{\rm 2N},\, \boldsymbol{T}] \neq 0, \quad [V_{II}^{\rm 2N},\, P_{\rm cr}] = [V_{II}^{\rm 2N},\, (\boldsymbol{T})^2] = 0 \Longrightarrow \boxed{V_{II}^{\rm 2N} = \alpha\, \tau_1^3 \tau_2^3}$$

$P_{\rm cr} = \exp(i\pi T_2)$

Evidence: $1/2(\delta^\alpha_{\rm nn} + \delta^\alpha_{\rm pp,\, str}) \neq \delta^\alpha_{\rm np}$

In particular: $a^{1S0}_{\rm nn} \simeq -18.9$ fm, $a^{1S0}_{\rm pp,\, str} \simeq -17.5$ fm, $a^{1S0}_{\rm np} = -23.74(2)$ fm

Class III (charge symmetry breaking, no isospin mixing):

$$[V_{III}^{\rm 2N},\, \boldsymbol{T}] \neq 0, \quad [V_{III}^{\rm 2N},\, P_{\rm cr}] \neq 0, \quad [V_{III}^{\rm 2N},\, (\boldsymbol{T})^2] = 0 \Longrightarrow \boxed{V_{III}^{\rm 2N} = \alpha\, (\tau_1^3 + \tau_2^3)}$$

Evidence: $\delta^\alpha_{\rm nn} \neq \delta^\alpha_{\rm pp,\, str}$, BE difference of mirror nuclei, …

Class IV (charge symmetry breaking and isospin mixing):

$$[V_{IV}^{\rm 2N},\, \boldsymbol{T}] \neq 0, \quad [V_{IV}^{\rm 2N},\, P_{\rm cr}] \neq 0, \quad [V_{IV}^{\rm 2N},\, (\boldsymbol{T})^2] \neq 0 \Longrightarrow \boxed{V_{IV}^{\rm 2N} = \alpha\, (\tau_1^3 - \tau_2^3) + \beta[\boldsymbol{\tau}_1 \times \boldsymbol{\tau}_2]^3}$$

Evidence: different neutron/proton analyzing powers in np scattering, …

From potential to phase shifts

Nonrelativistic Lippmann-Schwinger (LS) equation in partial waves (finite-range potential)

$$T^{sj}_{l'l}(p',p) = V^{sj}_{l'l}(p',p) + \sum_{\tilde{l}} \int_0^\infty \frac{d\tilde{p}\,\tilde{p}^2}{(2\pi)^3}\, V^{sj}_{l'\tilde{l}}(p',\tilde{p}) \frac{m}{p^2 - \tilde{p}^2 + i\eta} T^{sj}_{\tilde{l}l}(\tilde{p},p)$$

where $V^{sj}_{l'l}(p',p) \equiv \underbrace{\langle p',\, l'sj|\hat{V}|p,\, lsj\rangle}_{\text{standard expressions, available for all possible operators in } V}$ and $T^{sj}_{l'l}(p',p) = \underbrace{T^{sj}_{l'l}(p',p,k)\Big|_{k=p}}_{\text{half-off-shell T-matrix}} \equiv \langle p',\, l'sj|\hat{T}(k)|p,\, lsj\rangle\Big|_{k=p}$

- Uncoupled: $s=0,1,\;\; l=l'=j$, e. g. $^1S_0$, $^1P_1$, $^3P_1$, …, and $s=1,\; l=l'=1,\; j=0$ ($^3P_0$)
- Coupled: $s=1,\;\; l,l'=j\pm 1$, e. g. $^3S_1$-$^3D_1$ ⟹ LS equation is a 2 x 2 matrix equation

Once LS equation is solved using standard methods, phase shifts can be obtained as follows:

$$S^{sj}_{l'l}(k) = \delta_{l'l} - \frac{i}{8\pi^2}\, k\, m\, T^{sj}_{l'l}(k,k,k) \Longrightarrow \begin{cases} S^{sj}_{ll} = e^{2i\delta} \quad \textit{in the uncoupled case} \\ \begin{pmatrix} S^{1j}_{--} & S^{1j}_{-+} \\ S^{1j}_{+-} & S^{1j}_{++} \end{pmatrix} = \begin{pmatrix} e^{2i\delta_-}\cos 2\epsilon & i e^{i(\delta_-+\delta_+)}\sin 2\epsilon \\ i e^{i(\delta_-+\delta_+)}\sin 2\epsilon & e^{2i\delta_+}\cos 2\epsilon \end{pmatrix} \\ \textit{(Stapp parametrization in the coupled case)} \end{cases}$$

Once S-matrix is known, all NN scattering observables can be calculated straightforwardly.

Bystricky, Lehar, Winternitz, J. Phys. (Paris) 39 (78) 1, La France, Winternitz, J. Phys. (Paris) 41 (80) 1391

Long-range electromagnetic interactions

Electromagnetic interaction between point-like nucleons up to and including $\mathcal{O}(\alpha^2)$ - and $\mathcal{O}(1/m_N^2)$ -terms:

$$V_{\rm EM}(pp) = V_{\rm C}^{\rm improved} + V_{\rm VP} + V_{\rm MM}(pp)\,, \qquad V_{\rm EM}(np) = V_{\rm MM}(np)\,, \qquad V_{\rm EM}(nn) = V_{\rm MM}(nn)$$

- **Improved Coulomb potential (leading $1/m_N{}^2$-corrections to 1γ + 2γ-exchange)**
  Austin, de Swart '83

$$V_{\rm C}^{\rm improved} = \frac{\alpha'}{r}\left(1 - \frac{\alpha}{m_p r}\right) \quad \text{with} \quad \alpha' = \alpha \frac{m_p^2 + 2k^2}{m_p\sqrt{m_p^2 + k^2}}$$

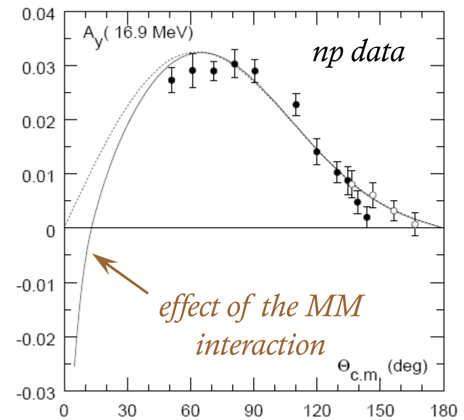
scattering (on-shell) momentum

- **Vacuum polarization**
  Ueling '35, Durand III '57

$$V_{\rm VP} = \frac{2\alpha}{3\pi}\frac{\alpha'}{r}\int_1^\infty dx\, e^{-2m_e r x}\left(1 + \frac{1}{2x^2}\right)\frac{(x^2-1)^{1/2}}{x^2}\,,$$

- **Magnetic moment interaction**
  Schwinger'48; Breit'55,'62; Stoks, de Swart, PRC 42 (1990) 1235

$$\begin{aligned}
V_{\rm MM}(pp) &= -\frac{\alpha}{4m_p^2 r^3}\left[\mu_p^2 S_{12} + (6 + 8\kappa_p)\vec{L}\cdot\vec{S}\right]\,,\\
V_{\rm MM}(np) &= -\frac{\alpha\kappa_n}{2m_n r^3}\left[\frac{\mu_p}{2m_p}S_{12} + \frac{1}{m}\left(\vec{L}\cdot\vec{S} + \frac{1}{2}\vec{L}\cdot(\vec{\sigma}_1 - \vec{\sigma}_2)\right)\right]\,,\\
V_{\rm MM}(nn) &= -\frac{\alpha\mu_n^2}{4m_n^2 r^3}S_{12}
\end{aligned}$$



NN force: a phenomenological approach

Strategy: take into account the known longest-range physics due to EM force and 1π-exchange

$V_{1\pi}(\vec{q}) \propto \frac{\vec{\sigma}_1 \cdot \vec{q}\, \vec{\sigma}_2 \cdot \vec{q}}{\vec{q}^{\,2} + M_\pi^2}\, \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2$ or in r-space: $V_{1\pi}^{\rm long}(\vec{r}) \propto \frac{e^{-M_\pi r}}{r} \left[S_{12} \left(1 + \frac{3}{M_\pi r} + \frac{3}{(M_\pi r)^2} \right) + \vec{\sigma}_1 \cdot \vec{\sigma}_2 \right] \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2$

and parametrize the medium- and short-range contributions in a most general way.

Example: AV18 potential

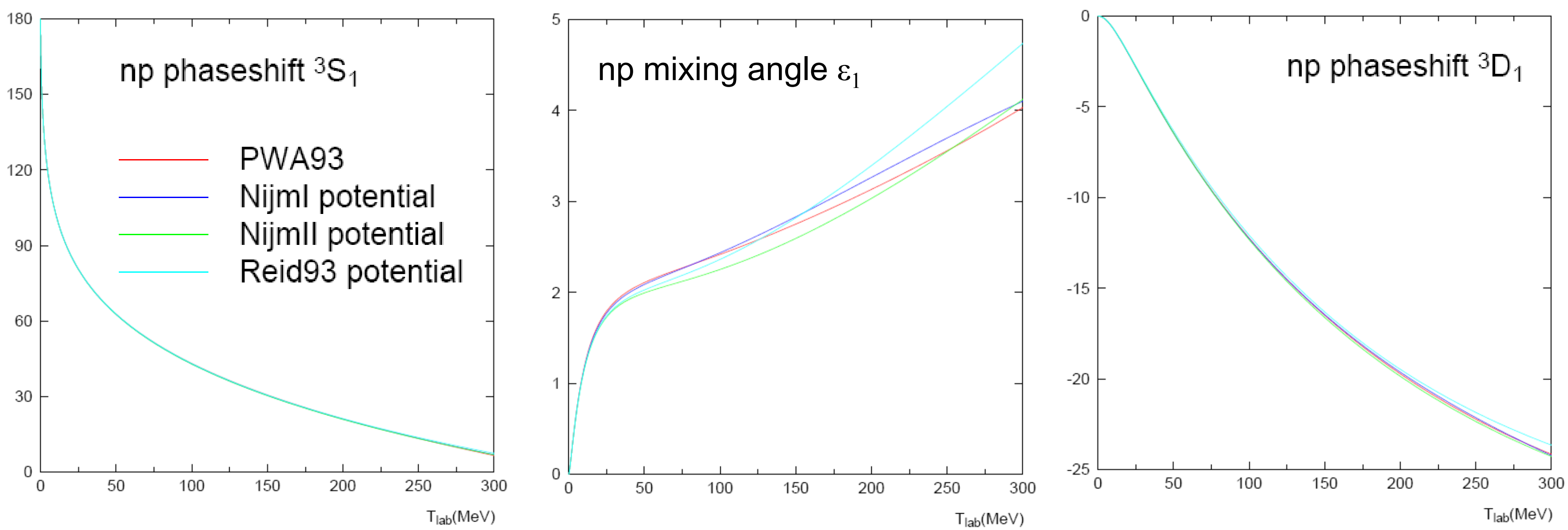
Wiringa, Stoks, Schiavilla '94

- Local r-space potential
- EM contributions multiplied by short-range functions to account the finite size of the nucleons
- Regularized OPEP including isospin breaking due to $M_{\pi^\pm} \neq M_{\pi^0}$
- Some additional phenomenological shorter range isospin-breaking terms
- Medium-range ($r \sim (2M_\pi)^{-1}$) contributions of Yukawa-type, short-range ones of the Woods-Saxon type
- 40 adjustable parameters fitted to 4301 pp and np scattering data, $\chi^2_{\rm datum} = 1.09$

Phenomenological NN potentials

Other phenomenological potentials

- OBE motivated nonlocal (Nijm I, 41 parameters; CD Bonn, 43 parameters) and local (Nijm II, 47 parameters; Reid93, 50 parameters) potentials. All have $\chi^2_{\rm datum} \sim 1$ and are define in the partial wave basis.
- BE models (Nijm93, Bonn): less parameters but higher $\chi^2_{\rm datum}$
- INOY, CD Bonn + Δ, inverse scattering, $V_{\text{low-k}}$, …



phase shifts generated by http://nn-online.org/

Three nucleons

JUNE 15, 1939 PHYSICAL REVIEW VOLUME 55

Many-Body Interactions in Atomic and Nuclear Systems

H. PRIMAKOFF, *Polytechnic Institute of Brooklyn, Brooklyn, New York*

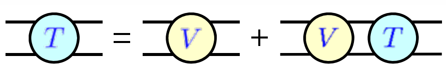
AND

T. HOLSTEIN,* *New York University, University Heights, New York, New York*

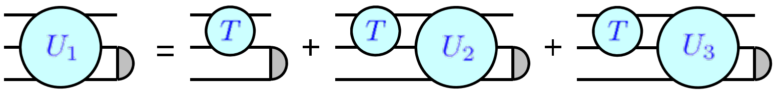
(Received March 28, 1938)

„…replacement of field interactions by two-body action-at-a-distance potentials is a poor approximation in nuclear physics.“

- Lippmann-Schwinger equation



- Set of three coupled integral equations



Faddeev ’65; Alt, Grassberger, Sandhas ’67,‘72

3N calculations based on phenomenological NN potentials show evidence for missing 3N forces (e.g. the underbinding of ^{3}H by about 1 MeV).

Three nucleons

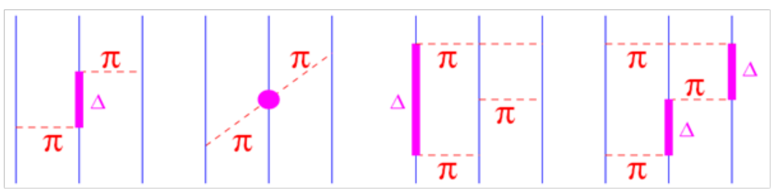
Most general parametrization of the 3NF seems not feasible:

- too many possible structures (> 100)
- too scarce data base available
- too involved calculations

⇨ need guidance from theory

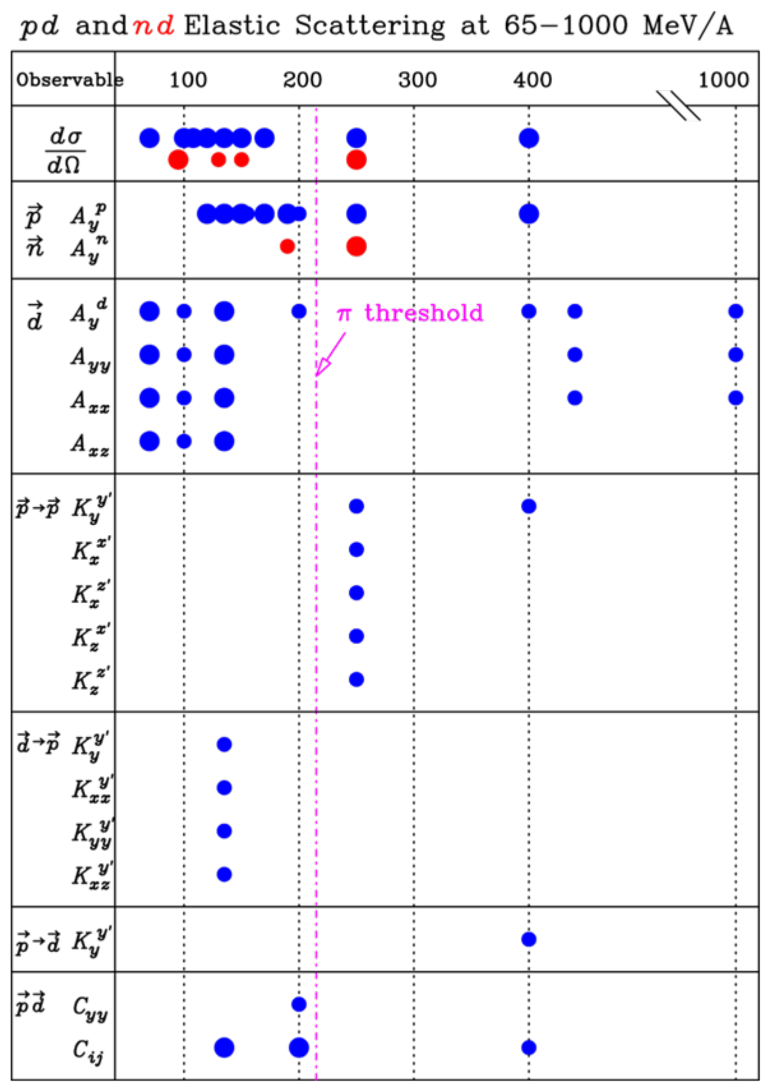
Three-nucleon force models

Fujita-Miyazawa, Brazil, Tucson-Melbourne, Urbana IX, Illinois, …



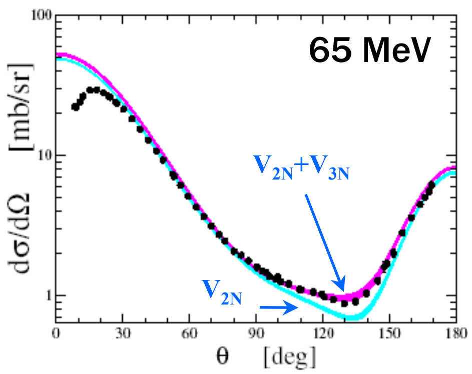
The strategy:

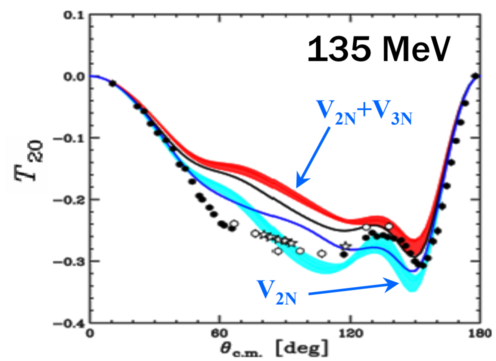
Take various combinations of 2N and 3N potentials, adjust parameters of the 3NF model to reproduce e.g. ^{3}H BE and apply resulting $V_{2N} + V_{3N}$ to scatt. observables.



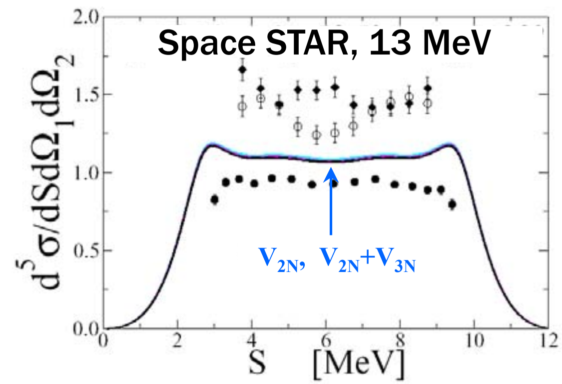
Successes and failures

Elastic scattering observables

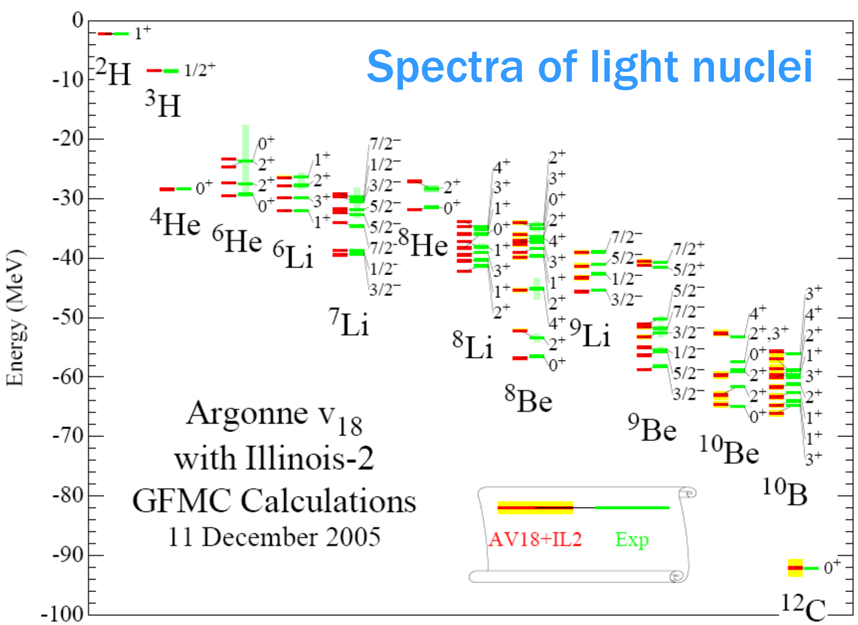




Deuteron breakup



Spectra of light nuclei



Inclusion of the 3NF sometimes leads to improvements, sometimes — not. Situation, in part, chaotic.

Need a theoretical approach which would:

- be based on QCD,
- yield consistent many-body forces,
- be systematically improvable,
- allow for error estimation

⇨ chiral effective field theory

Further reading

Some modern "high-precision" nucleon-nucleon potentials

- *Stoks, Klomp, Terheggen, de Swart, Phys. Rev. C49 (1994) 2950   [Nijmegen 93, Nijm I,II, Reid 93]*
- *Wiringa, Stoks, Schiavilla, Phys. Rec. C51 (95) 38   [Argonne V18]*
- *Machleidt, Phys. Rev. C63 (01) 024001   [CD Bonn 2000]*
- *Machleidt, Slaus, J. Phys. G27 (01) R69   [review article]*

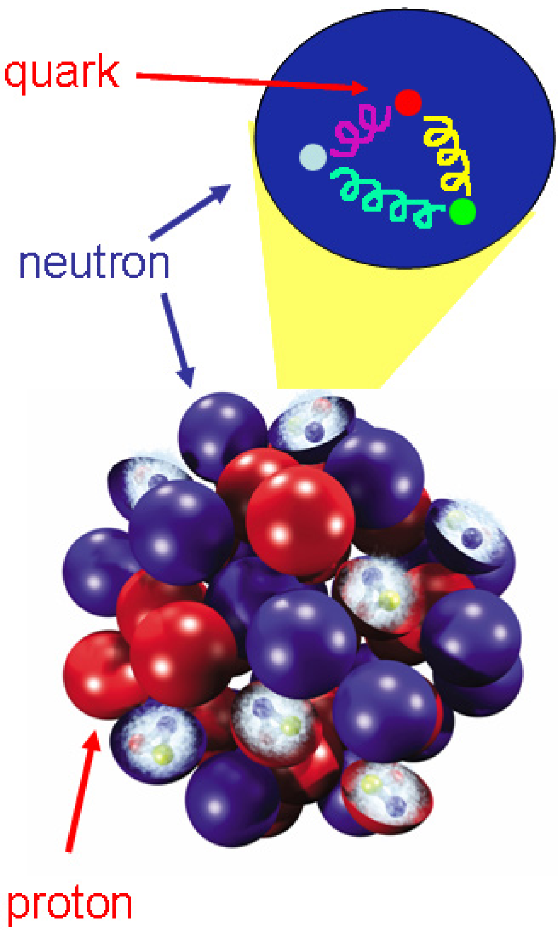
Three-nucleon force models

- *Fujita, Miyazawa, Prog. Theor. Phys. 17 (57) 360   [Fujita-Miyazawa 3NF model]*
- *Coon, Han, Few-Body Syst. 30 (01) 131   [Tucson-Melbourne 3NF model]*
- *Coelho, Das, Robilotta, Phys. Rev. C28 (83) 1812   [Brazilian 3NF model]*
- *Pudliner, Pandharipande, Carlson, Pieper, Wiringa, Phys. Rev. C56 (97) 1720   [Urbana IX 3NF model]*
- *Pieper, Wiringa, Ann. Rev. Nucl. Part. Sci. 51 (01) 53   [Illinois 3NF model]*

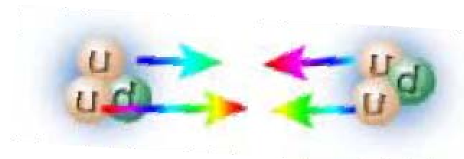
Review articles on 3N scattering & 3N force effects

- *Glöckle, Witala, Huber, Kamada, Golak, Phys. Rept. 274 (96) 107*
- *Kalantar-Nayestanaki, E.E., Nucl. Phys. News 17 (07) 22*

From QCD to nuclear forces

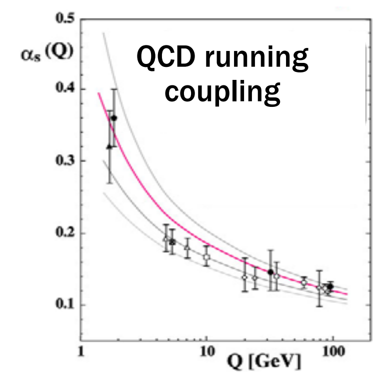


- QCD: nuclear force is due to residual color force



However…

- non-perturbative at low energy
- "wrong" degrees of freedom



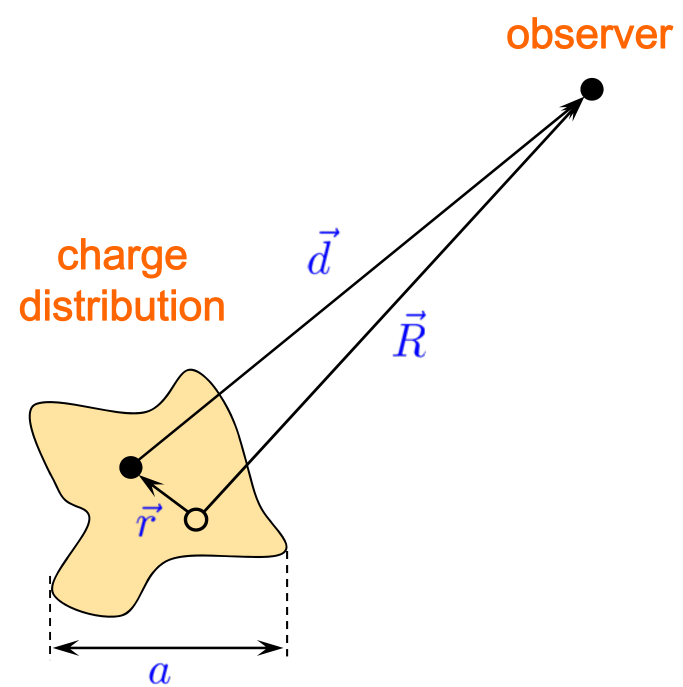
- Nonperturbative methods
  - — lattice QCD
    Nemura, Ishii, Aoki, Detmold, ...
  - — effective field theory
    Weinberg, ...
  - — large-N_c expansion
    Kaplan, Savage, Cohen, ...

Effective field theories

An effective (field) theory is an approximate theory whose scope is to describe phenomena which occur at a chosen length (or energy) range.

Example: multipole expansion for electric potentials

$$
\begin{aligned}
V &\propto \int \frac{\rho(\vec{r})}{d}\, d^3r \\
&= \int \frac{\rho(\vec{r})}{\sqrt{R^2 + 2rR\cos\theta + r^2}}\, d^3r \\
&= \sum_{n=0}^{\infty} \frac{1}{R^{n+1}} \int r^n\, P_n(\cos\theta)\rho(\vec{r})\, d^3r \\
&= \underbrace{q\frac{1}{R} + P\frac{1}{R^2} + Q\frac{1}{R^3} + \dots}_{\textit{the sum converges rapidly for } a \ll R}
\end{aligned}
$$



Weinberg's theorem

"if one writes down the most general possible Lagrangian, including all terms consistent with the assumed symmetry principles, and then calculates S-matrix elements with this Lagrangian to any order in perturbation theory, the result will simply be the most general possible S-matrix consistent with analyticity, perturbative unitarity, cluster decomposition and the assumed symmetry principles"

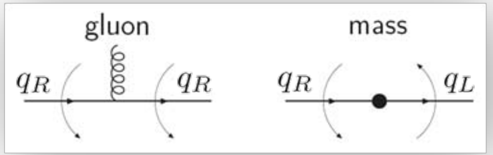
S.Weinberg, Physica A96 (79) 327

- identify the symmetries of the underlying theory,
- construct the most general $\mathcal{L}_{\rm eff}$ in terms of relevant d.o.f. and consistent with the symmetries,
- do standard quantum field theory with the effective Lagrangian.

Chiral symmetry of QCD Lagrangian

$$\mathcal{L}_{\rm QCD} = -\tfrac{1}{4}G_{\mu\nu}G^{\mu\nu} + \bar{q}(i\not{D} - \mathcal{M})q \ = \underbrace{-\tfrac{1}{4}G_{\mu\nu}G^{\mu\nu} + \bar{q}_L i\not{D} q_L + \bar{q}_R i\not{D} q_R}_{SU(2)_L \times SU(2)_R \ \textit{invariant}} - \underbrace{\bar{q}_L \mathcal{M} q_R - \bar{q}_R \mathcal{M} q_L}_{\textit{breaks chiral symmetry}}$$

Left- and right-handed quark fields: $q_{L,R} = \frac{1}{2}(1 \pm \gamma_5)q.$



Chiral group is a group of independent rotations of $q_{L,\,R}$ in the flavor space.

For 2 flavors: $\mathrm{G = SU(2)_L \times SU(2)_R}$ and $\begin{cases} q_L \xrightarrow{G} q'_L = g_L\, q_L \\ q_R \xrightarrow{G} q'_R = g_R\, q_R \end{cases}$ with $g_{L,R} \in \mathrm{SU}(2)_{L,R}$

Chiral SU(2) Lie algebra:

$$\begin{aligned} [\Gamma^L_i,\ \Gamma^L_j] &= i\epsilon_{ijk}\Gamma^L_k \\ [\Gamma^R_i,\ \Gamma^R_j] &= i\,\epsilon_{ijk}\,\Gamma^R_k \\ [\Gamma^L_i,\ \Gamma^R_j] &= 0 \end{aligned}$$

or

$$\begin{aligned} [V_i,\ V_j] &= i\,\epsilon_{ijk}\,V_k \\ [A_i,\ A_j] &= i\,\epsilon_{ijk}\,V_k \\ [V_i,\ A_j] &= i\,\epsilon_{ijk}\,A_k \end{aligned}$$

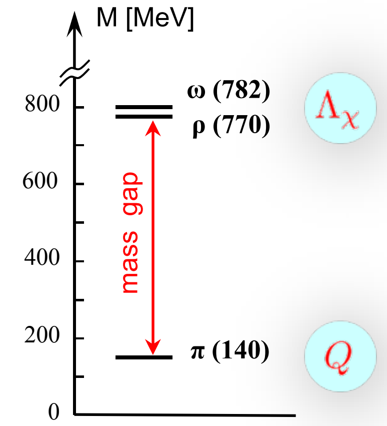
with $\underbrace{V_i = \Gamma^R_i + \Gamma^L_i}_{\textit{vector (isospin) generators}}, \ \underbrace{A_i = \Gamma^R_i - \Gamma^L_i}_{\textit{axial generators}}$

$m_{u,\,d}$ small $\Longrightarrow$ $\mathcal{L}_{\rm QCD}$ is approximately ($M^2_\pi / M^2_\rho \sim 0.03$) chiral invariant

Chiral symmetry of QCD

There is a strong evidence that chiral symmetry of QCD is spontaneously broken down to the isospin group:

- Only isospin but not chiral multiplets are observed in the particle spectrum (axial charges would lead to parity doublets)



- Triplet of unnaturally light pseudoscalar mesons (pions) — natural candidates for Goldstone bosons

- Scalar quark condensate:

$$\langle 0|\bar{q}q|0\rangle\Big|_{\overline{MS},\,2\,GeV} = -(273 \pm 12\ \text{MeV})^3$$

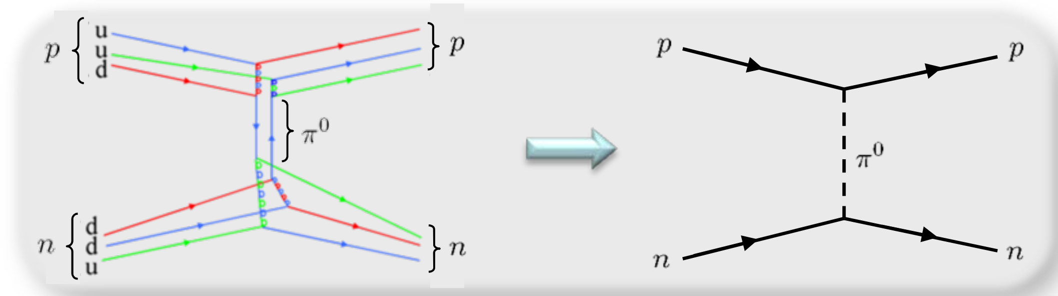
(Lattice QCDSF/UKQCD, *Schierholz et al. '07*)

- Further theoretical arguments
Vafa & Witten '84; 't Hooft '80; Coleman & Witten '80

Chiral Perturbation Theory

Weinberg, Gasser, Leutwyler, Bernard, Kaiser, Meißner, …

- asymptotically observed states as effective DOF → EFT
- spontaneously broken approximate χ-symmetry of QCD plays a crucial role
- light (M_π) and heavy (M_ρ) scales well separated



$$\int [Dq][D\bar{q}][DG]\; e^{\int id^4x\,\mathcal{L}_{\rm QCD}[q,\bar{q},G;\,J]} \Longrightarrow \int [D\Phi]\; e^{\int id^4x\,\mathcal{L}_{\rm eff}[\Phi;\,J]}$$

Cannot derive $\mathcal{L}_{eff}$ ⟹ write most general expression consistent with χ-symmetry, i.e.:

- include all possible χ-invariant terms,
- include all terms that break χ-symmetry in the same way as $\bar{q}mq$ in $\mathcal{L}_{\rm QCD}$ does.

Consider the pure Goldstone boson sector in the chiral limit.

- How to write down most general χ-invariant $\mathcal{L}_{eff}$?
- How do π's transform under G?
- Isospin subgroup $H \in G$ realized linearly (π's build an isospin triplet).
- Chiral group necessarily realized nonlinearly ($SU(2)_L \times SU(2)_R$ isomorphic to $SO(4)$
  ⟹ need at least 4 dimensions to construct a nontrivial linear realization)

Chiral rotations & pion fields

Infinitesimal $\mathrm{SO(4)}$ rotation of the 4-vector $(\pi_1,\, \pi_2,\, \pi_3,\, \sigma)$:

$$\begin{pmatrix} \vec{\pi} \\ \sigma \end{pmatrix} \to \begin{pmatrix} \vec{\pi}' \\ \sigma' \end{pmatrix} = \left[1 + \vec{\theta}^{\,V} \cdot \vec{V} + \vec{\theta}^{\,A} \cdot \vec{A}\right] \begin{pmatrix} \vec{\pi} \\ \sigma \end{pmatrix}$$

where: $\vec{\theta}^{\,V} \cdot \vec{V} = \begin{pmatrix} 0 & -\theta_3^V & \theta_2^V & 0 \\ \theta_3^V & 0 & -\theta_1^V & 0 \\ -\theta_2^V & \theta_1^V & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $\vec{\theta}^{\,A} \cdot \vec{A} = \begin{pmatrix} 0 & 0 & 0 & \theta_1^A \\ 0 & 0 & 0 & \theta_2^A \\ 0 & 0 & 0 & \theta_3^A \\ -\theta_1^A & -\theta_2^A & -\theta_3^A & 0 \end{pmatrix}$

One reads off: $\vec{\pi}\,' = \vec{\pi} + \vec{\theta}^{\,V} \times \vec{\pi} + \vec{\theta}^{\,A} \sigma$ and $\sigma' = \sigma - \vec{\theta}^{\,A} \cdot \vec{\pi}$

Switch to the nonlinear realization of $\mathrm{SO(4)}$:
only 3 out of 4 components of the vector $(\vec{\pi},\, \sigma)$ are independent, i.e. $\vec{\pi}^2 + \sigma^2 = F^2$

$$\sigma = \sqrt{F^2 - \vec{\pi}^2} \quad \Rightarrow \quad \begin{cases} \vec{\pi} \xrightarrow{\vec{\theta}^{\,V}} \vec{\pi}' = \vec{\pi} + \vec{\theta}^{\,V} \times \vec{\pi} & \longleftarrow \textit{linear under } \vec{\theta}^{\,V} \\ \vec{\pi} \xrightarrow{\vec{\theta}^{\,A}} \vec{\pi}' = \vec{\pi} + \vec{\theta}^{\,A} \sqrt{F^2 - \vec{\pi}^2} & \longleftarrow \textit{nonlinear under } \vec{\theta}^{\,A} \end{cases}$$

It is more convenient to use a 2 x 2 matrix notation:

$$U = \tfrac{1}{F}\left(\sigma\, I + i\vec{\pi} \cdot \vec{\tau}\right) \xrightarrow[\textit{realization}]{\textit{nonlinear}} U = \tfrac{1}{F}\left(I\sqrt{1 - \vec{\pi}^{\,2}} + i\vec{\pi} \cdot \vec{\tau}\right)$$

Chiral rotations: $U \longrightarrow U' = L U R^\dagger$ with $L = e^{-i/2(\vec{\theta}^{\,V} - \vec{\theta}^{\,A}) \cdot \vec{\tau}}$ and $R = e^{-i/2(\vec{\theta}^{\,V} + \vec{\theta}^{\,A}) \cdot \vec{\tau}}$

Effective Lagrangian

The above realization of G is not unique. How does this non-uniqueness affect S-matrix?

- All realizations of G are equivalent to each other by means of nonlinear field redefinitions $\vec{\pi} \to \vec{\pi}\,' = \vec{\pi}\, F[\vec{\pi}\,],\ F[0] = 1$ *(Coleman, Callan, Wess & Zumino '69)*
- Such field redefinitions do not affect S-matrix *(Haag '58)*

Derivative expansion for the effective Lagrangian $\mathcal{L}_{\rm eff} = \mathcal{L}_\pi^{(2)} + \mathcal{L}_\pi^{(4)} + \ldots$

- 0 derivatives: $UU^\dagger = U^\dagger U = 1$ — plays no role
- 2 derivatives: $\mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger) \xrightarrow{g\in G} \mathrm{Tr}(L\partial_\mu U R^\dagger R\, \partial^\mu U^\dagger L^\dagger) = \mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger)$

  $$\Longrightarrow \quad \mathcal{L}_\pi^{(2)} = \frac{F^2}{4}\mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger)$$

  derivatives act only on the next U
- 4 derivatives: $[\mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger)]^2\,,\ \mathrm{Tr}(\partial_\mu U \partial_\nu U^\dagger)\mathrm{Tr}(\partial^\mu U \partial^\nu U^\dagger)\,,\ \mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger \partial_\nu U \partial^\nu U^\dagger)$

  (terms with $\partial_\mu\partial_\nu U,\ \partial_\mu\partial_\nu\partial_\rho U,\ \partial_\mu\partial_\nu\partial_\rho\partial_\sigma U$ can be eliminated via EOM/partial integration)

...

What is the meaning of F ?

Axial current from $\mathcal{L}_\pi^{(2)}$: $J^i_{A\mu} = i\mathrm{Tr}[\tau^i(U^\dagger\partial_\mu U - U\partial_\mu U^\dagger)] = -F\partial_\mu \pi^i + \ldots$ *more pion fields*

$$\langle 0 | J^i_{A\mu} | \pi^j(\vec{p}\,)\rangle \equiv i p_\mu F_\pi \delta^{ij} \quad \Longrightarrow \quad F = F_\pi = 92.4\ \mathrm{MeV}$$

Effective Lagrangian

How to account for explicit χ-symmetry breaking due to nonvanishing quark masses?

external hermitian field

Trick (method of external sources): $\delta\mathcal{L}_{\rm QCD} = -\bar{q}\mathcal{M}q\Big|_{\mathcal{M}=m}$

$-\bar{q}\mathcal{M}q = -\bar{q}_L\mathcal{M}q_R - \bar{q}_R\mathcal{M}q_L$ is χ -invariant if: $\mathcal{M} \xrightarrow{G} \mathcal{M}' = g_R\mathcal{M}g_L^{-1} = g_L\mathcal{M}g_R^{-1}$

⇒ write down all possible χ–invariant terms with $\mathcal{M}$ and then set $\mathcal{M} = m$

The leading (i.e. no ∂_μ and $\propto \mathcal{M}$) SB term in $\mathcal{L}_{eff}$:

$$\mathcal{L}_{\rm SB} = \frac{BF^2}{2}{\rm Tr}[(U+U^\dagger)\mathcal{M}]\Big|_{\mathcal{M}=m} = 2BF^2 m_q - B\, m_q\, \vec{\pi}^{\,2} + \mathcal{O}(\vec{\pi}^{\,4}) \quad \Longrightarrow \quad M_\pi^2 = 2m_q B + \mathcal{O}(m_q^2)$$

The LEC B is related to the scalar quark condensate via $\langle 0|\bar{u}u|0\rangle = \langle 0|\bar{d}d|0\rangle = -BF^2 + \mathcal{O}(\mathcal{M})$

Notice: the generalized scenario (***Stern et al. '91***) in which $2m_q B \ll M_\pi^2$ is ruled out by recent data on $\pi\pi$ scatt. length.

Effective Lagrangian

Gasser, Leutwyler, Nucl. Phys. B250 (1985) 465

low-energy constants

$$\mathcal{L}_\pi^{(2)} = \frac{F^2}{4}\left[\mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger) + \mathrm{Tr}(U\chi + U^\dagger\chi)\right],$$

$$\begin{aligned}\mathcal{L}_\pi^{(4)} &= L_1[\mathrm{Tr}(\partial_\mu U^\dagger \partial^\mu U)]^2 + L_2 \mathrm{Tr}(\partial_\mu U^\dagger \partial_\nu U)\mathrm{Tr}(\partial^\mu U^\dagger \partial^\nu U) + L_3 \mathrm{Tr}(\partial_\mu U^\dagger \partial^\mu U \partial_\nu U^\dagger \partial^\nu U) \\ &+ L_4 \mathrm{Tr}(\partial_\mu U^\dagger \partial^\mu U)\mathrm{Tr}(U\chi + U^\dagger\chi) + L_5 \mathrm{Tr}(\partial_\mu U^\dagger \partial^\mu U(U\chi + U^\dagger\chi)) + L_6[\mathrm{Tr}(U\chi + U^\dagger\chi)]^2 \\ &+ L_7[\mathrm{Tr}(U\chi - U^\dagger\chi)]^2 + L_8 \mathrm{Tr}(\chi U \chi U + \chi U^\dagger \chi U^\dagger)\end{aligned}$$

where $\chi = 2B\mathcal{M}$.

- Only those terms are shown which do not involve external sources (there are 3 more terms which describe the interaction of GBs with external fields).
- The Lagrangian is shown for the SU(3) x SU(3) case. Some terms are redundant in the case of SU(2) x SU(2) chiral symmetry.
- How to calculate observables ??

→ see next lecture…