SYMMETRIES IN NUCLEI

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Abstract. The use of dynamical symmetries or spectrum generating algebras for the solution of the nuclear many-body problem is reviewed. General notions of symmetry and dynamical symmetry in quantum mechanics are introduced and illustrated with simple examples such as U(3) symmetry of the three-dimensional harmonic oscillator. The nuclear shell model is reviewed with particular emphasis on the use of group-theoretical techniques in the context of this model.

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1. INTRODUCTION

In the *Oxford Dictionary of Current English* symmetry is defined as the 'right correspondence of parts; quality of harmony or balance (in size, design etc.) between parts'. The word is derived from Greek where it has the meaning 'with proportion' or 'with order'. In modern theories of physics it has acquired a more precise meaning but the general idea of seeking to order physical phenomena still remains. Confronted with the bewildering complexity exhibited by the multitude of physical systems, physicists attempt to extract some simple regularities from observations, and the fact that they can do so is largely due to the presence of symmetries in the laws of physics. Although one can never hope to explain all observational complexities on the basis of symmetry arguments alone, these are nevertheless instrumental in establishing correlations between and (hidden) regularities in the data.

The mathematical theory of symmetry is called group theory and its origin dates back to the nineteenth century. Of course, the notion of symmetry is present implicitly in many mathematical studies that predate the birth of group theory and goes back even to the ancient Greeks, in particular Euclid. It was, however, Évariste Galois who perceived the importance of the group of permutations to answer the question whether the roots of a polynomial equation can be algebraically represented or not. (A readable summary of the solution of this problem is given in the first chapter of Gilmore's book [1].) In the process of solving that long-standing mathematical problem he invented group theory as well as Galois theory which studies the relation between polynomials and groups. The mathematical theory of groups developed further throughout the nineteenth century and made another leap forward in 1873 when Sophus Lie proposed the concept of a Lie group and its associated Lie algebra.

For a long time it was assumed that group theory was a branch of mathematics without any application in the physical sciences. This state of affairs changed with the advent of quantum mechanics, and it became clear that group theory provides a powerful tool

to understand the structure of quantum systems from a unified perspective. After the introduction of symmetry transformations in abstract spaces (associated, for example, with isospin, flavour, colour, etc.) the role of group theory became even central.

The purpose of these lecture notes is to introduce, explain and illustrate the concepts of symmetry and dynamical symmetry. In Sect. 2 a brief reminder is given of the central role of symmetry in quantum mechanics and of its relation with invariance and degeneracy. There exist two standard examples to illustrate the idea that symmetry implies degeneracy and *vice versa*, namely the hydrogen atom and the harmonic oscillator. In Sect. 3 the second of them is analyzed in detail. Section 4 describes the process of symmetry breaking and, in particular, dynamical symmetry breaking in the sense as it is used in these lecture notes. Section 5 presents the nuclear shell model with a special emphasis on the symmetry techniques that have been used in the context of this model. Finally, in Sect. 6, a summary is given.

These notes accompany two lectures given at the Joliot–Curie school on "Symmetries in Subatomic Systems" which was held in Lacanau, France, from 27 September to 3 October 2010. Similar lecture notes of mine appeared under the same title [2] at the occasion of the XL$^{\text{th}}$ Latin-American school on "Symmetry in Physics in Memoriam of Marcos Moshinsky" which was held in México D.F., Mexico, from 26 July to 6 August 2010. I have tried to minimize the overlap in content between the two contributions but some repetition of material seemed unavoidable. In particular, the basic notions of symmetry and dynamical symmetry, Sects. 2 and 4, and an elementary introduction to the nuclear shell model, Subsect. 5.1, needed to be included in both lecture notes.

A part from these similarities, the material presented here is different from the one in Ref. [2]. On both occasions isospin in nuclei was central to the lectures. Since a detailed discussion of this topic can be found in the previous lecture notes, it has been omitted from the present written account. The classic example of 'accidental' degeneracy discussed in the present lecture notes is the U(3) symmetry of the three-dimensional harmonic oscillator, Sect. 3, while in Ref. [2] it is the SO(4) symmetry of the hydrogen atom. A beautiful example of the application of symmetry techniques in nuclear physics is the interacting boson model of Arima and Iachello [3]. It is nevertheless omitted entirely here since this topic was covered at the Joliot–Curie school in the lectures of R.F. Casten. The interested reader may also consult my lecture notes at the XL$^{\text{th}}$ Latin-American school for an introduction to the interacting boson model. Finally, while the symmetry aspects of the nuclear shell model are only summarily treated in Ref. [2], they are central here and discussed at length.

2. SYMMETRY IN QUANTUM MECHANICS

The starting point of any discussion of symmetry is that the laws of physics should be invariant with respect to certain transformations of the reference frame, such as a translation or rotation, or a different choice of the origin of the time coordinate. This observation leads to three fundamental conservation laws: conservation of linear momentum, angular momentum and energy. In some cases an additional space-inversion symmetry applies, yielding another conserved quantity, namely parity. In a relativistic framework the above transformations on space and time cannot be considered separately

but become intertwined. The laws of nature are then invariant under the Lorentz transformations which operate in four-dimensional space–time.

These transformations and their associated invariances can be called 'geometric' in the sense that they are defined in space–time. In quantum mechanics, an important extension of these concepts is obtained by also considering transformations that act in abstract spaces associated with intrinsic variables such as spin, isospin (in atomic nuclei), flavour and colour (of quarks) etc. It is precisely these 'intrinsic' invariances which have lead to the preponderance of symmetry applications in the quantum physics.

To be more explicit, consider a transformation acting on a physical system, that is, an operation that transforms the coordinates $\bar{r}_i$ and the momenta $\bar{p}_i$ of the particles that constitute the system. Such transformations are of a geometric nature. For a discussion of symmetry in quantum-mechanical systems this definition is too restrictive and the appropriate generalization is to consider, instead of the geometric transformations themselves, the corresponding transformations in the Hilbert space of quantum-mechanical states of the system. The action of the geometric transformation on spin variables (*i.e.*, components of the spin vector) is assumed to be identical to its action on the components of the angular momentum vector $\bar{\ell} = \bar{r} \wedge \bar{p}$. Furthermore, it can be shown [4] that a correspondence exists between the geometric transformations in physical space and the transformations induced by it in the Hilbert space of quantum-mechanical states. This correspondence, however, is not necessarily one-to-one; that is only the case if the system is 'bosonic' (consists of any number of integer-spin bosons and/or an even number of half-integer-spin fermions). If the system is 'fermionic' (contains an odd number of fermions), the correspondence is two-to-one and the groups, formed by the geometric transformations and by the corresponding transformations in the Hilbert space of quantum-mechanical states, are not isomorphic but rather homomorphic.

No distinction is made in the following between geometric and quantum-mechanical transformations; all elements g_i will be taken as operators acting on the Hilbert space of quantum-mechanical states.

2.1. Symmetry

A time-independent hamiltonian H which commutes with the generators g_k that form a Lie algebra G,

$$\forall g_k \in \mathrm{G} : [H, g_k] = 0, \tag{1}$$

is said to have a symmetry G or, alternatively, to be invariant under G. The determination of operators g_k that leave invariant the hamiltonian of a given physical system is central to any quantum-mechanical description. The reasons for this are profound and can be understood from the correspondence between geometrical and quantum-mechanical transformations. It can be shown [4] that the transformations g_k with the symmetry property (1) are induced by geometrical transformations that leave unchanged the corresponding classical hamiltonian. In this way the classical notion of a conserved quantity is transcribed in quantum mechanics in the form of the symmetry property (1) of the time-independent hamiltonian.

2.2. Degeneracy and state labeling

A well-known consequence of a symmetry is the occurrence of degeneracies in the eigenspectrum of H. Given an eigenstate $|\gamma\rangle$ of H with energy E, the condition (1) implies that the states $g_k|\gamma\rangle$ all have the same energy,

$$Hg_k|\gamma\rangle = g_k H|\gamma\rangle = Eg_k|\gamma\rangle. \tag{2}$$

An arbitrary eigenstate of H shall be written as $|\Gamma\gamma\rangle$, where the first quantum number Γ is different for states with different energies and the second quantum number γ is needed to label degenerate eigenstates. The eigenvalues of a hamiltonian that satisfies (1) depend on Γ only,

$$H|\Gamma\gamma\rangle = E(\Gamma)|\Gamma\gamma\rangle, \tag{3}$$

and, furthermore, the transformations g_k do not admix states with different Γ,

$$g_k|\Gamma\gamma\rangle = \sum_{\gamma'} a^{\Gamma}_{\gamma'\gamma}(k)|\Gamma\gamma'\rangle. \tag{4}$$

This simple discussion of the consequences of a hamiltonian symmetry illustrates the relevance of group theory in quantum mechanics. Symmetry implies degeneracy and eigenstates that are degenerate in energy provide a Hilbert space in which irreducible representations of the symmetry group are constructed. Consequently, the irreducible representations of a given group directly determine the degeneracy structure of a hamiltonian with the symmetry associated to that group.

Eigenstates of H can be denoted as $|\Gamma\gamma\rangle$ where the symbol Γ labels the irreducible representations of G. Note that the same irreducible representation might occur more than once in the eigenspectrum of H and, therefore, an additional multiplicity label η should be introduced to define a complete labeling of eigenstates as $|\eta\Gamma\gamma\rangle$. This label shall be omitted in the subsequent discussion.

A sufficient condition for a hamiltonian to have the symmetry property (1) is that it is a Casimir operator which by definition commutes with all generators of the algebra. The eigenequation (3) then becomes

$$C_m[\mathrm{G}]|\Gamma\gamma\rangle = E_m(\Gamma)|\Gamma\gamma\rangle, \tag{5}$$

where $C_m[\mathrm{G}]$ is the mth order Casimir operator of the algebra G. In fact, all results remain valid if the hamiltonian is an analytic function of Casimir operators of various orders. The energy eigenvalues $E_m(\Gamma)$ are functions of the labels that specify the irreducible representation Γ, and are known for all classical Lie algebras [5].

3. THE HARMONIC OSCILLATOR IN THREE DIMENSIONS

The potential which best mimics the nuclear mean-field potential and which can be solved exactly, is the harmonic-oscillator potential

$$V(r) = \frac{1}{2} m_{\mathrm{n}} \omega^2 r^2. \tag{6}$$

In Sect. 5 more details are given on the relation between this schematic potential and the mean-field potential as observed in nuclei. The purpose of the present section is the study of the harmonic oscillator from the perspective of symmetries. The potential is independent of the spin of the nucleon which leads to a two-fold degeneracy of all states corresponding to spin-up and spin-down. Spin is ignored in the following and the symmetry properties of the spatial part only of the nucleon wave functions are studied.

The eigensolutions of the Schrödinger equation of a harmonic oscillator in three dimensions can be written as [6]

$$\phi_{n\ell m_\ell}(r,\theta,\varphi) = R_{n\ell}(r) Y_{\ell m_\ell}(\theta,\varphi), \tag{7}$$

where $R_{n\ell}(r)$ are radial wave functions appropriate for the harmonic oscillator and $Y_{\ell m_\ell}(\theta,\varphi)$ are spherical harmonics. The quantized energy spectrum is given by

$$E(n,\ell) = \left(2n+\ell+\tfrac{3}{2}\right)\hbar\omega, \tag{8}$$

in terms of the radial quantum number n which has the allowed values $0,1,2,\ldots$ and gives the number of nodes [values of r for which $R_{n\ell}(r)=0$ excluding those at $r=0$ and $r=\infty$]. Because of the factor r^ℓ in the radial part, the wave function always vanishes at $r=0$ except for $\ell=0$ (s state). The energy $E(n,\ell)$ is independent of m_ℓ, the projection of the orbital angular momentum along the z axis, as should be for a rotationally invariant hamiltonian. In addition, $E(n,\ell)$ is only dependent on the sum $2n+\ell$. Introducing $N = 2n+\ell$, one can rewrite (8) as

$$E(N) = \left(N+\tfrac{3}{2}\right)\hbar\omega, \tag{9}$$

which shows that N can be interpreted as the number of oscillator quanta, the term $\frac{3}{2}\hbar\omega$ being accounted for by the zero-point motion of an oscillator in three dimensions; N is called the major oscillator quantum number. The allowed values of the orbital angular momentum are (because $\ell = N-2n$ and $n=0,1,\ldots$)

$$\ell = N, N-2, \ldots, 0 \text{ or } 1. \tag{10}$$

This completely determines the eigenspectrum of a spinless particle in a harmonic-oscillator potential.

The $2\ell+1$ eigensolutions with the same radial quantum number n and the same orbital angular momentum ℓ but different z projections m_ℓ are degenerate in energy. This degeneracy arises because the harmonic-oscillator hamiltonian is rotationally invariant. There exists an *additional* degeneracy, namely the one for levels with the same $2n+\ell$ which is illustrated in Fig. 1 for the lowest levels of the three-dimensional harmonic oscillator.

To understand the origin of this additional degeneracy, let us introduce the following raising and lowering operators [6]:

$$\begin{aligned} b_x^\dagger &= \frac{1}{\sqrt{2}}\left(x' - \frac{\partial}{\partial x'}\right), \quad b_y^\dagger = \frac{1}{\sqrt{2}}\left(y' - \frac{\partial}{\partial y'}\right), \quad b_z^\dagger = \frac{1}{\sqrt{2}}\left(z' - \frac{\partial}{\partial z'}\right), \\ b_x &= \frac{1}{\sqrt{2}}\left(x' + \frac{\partial}{\partial x'}\right), \quad b_y = \frac{1}{\sqrt{2}}\left(y' + \frac{\partial}{\partial y'}\right), \quad b_z = \frac{1}{\sqrt{2}}\left(z' + \frac{\partial}{\partial z'}\right), \end{aligned} \tag{11}$$

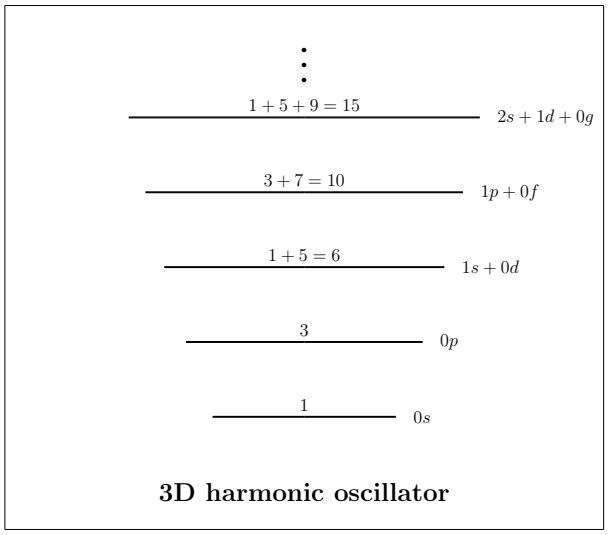

FIGURE 1. The energy spectrum of the harmonic oscillator in three dimensions.

where the primed coordinates are dimensionless, that is, $x' = x/b$, $y' = y/b$ and $z' = z/b$, with b the length parameter of the oscillator, $b = \sqrt{\hbar/m_{\rm n}\omega}$. The raising and lowering operators satisfy commutations rules appropriate for bosons, *viz.*

$$[b_i, b_j] = [b_i^\dagger, b_j^\dagger] = 0, \qquad [b_i, b_j^\dagger] = \delta_{ij}. \tag{12}$$

Furthermore, in terms of these operators, the hamiltonian of the harmonic oscillator can be written as

$$H_{\rm ho} = \frac{p^2}{2m_{\rm n}} + \frac{1}{2}m_{\rm n}\omega^2 r^2 = \sum_{i=x,y,z}\left(b_i^\dagger b_i + \frac{1}{2}\right)\hbar\omega. \tag{13}$$

Since this hamiltonian manifestly conserves the total number of bosons, it commutes with the nine bilinear combinations $u_{ij} \equiv b_i^\dagger b_j$,

$$\forall i, j \in x, y, z : [H_{\rm ho}, u_{ij}] = 0. \tag{14}$$

Finally, the nine operators u_{ij} can be shown to close under commutation,

$$[u_{ij}, u_{kl}] = u_{il}\delta_{jk} - u_{kj}\delta_{il}, \tag{15}$$

and the corresponding algebra can be identified as U(3). This establishes that the symmetry algebra of the harmonic oscillator in three dimensions is U(3). In fact, in an entirely analogous fashion it can be shown that the symmetry algebra of the harmonic oscillator in n dimensions is U(n).

How is this U(3) algebra related to the symmetry associated with rotational invariance? To answer this question, one rewrites the nine U(3) generators differently, in the following way:

$$\frac{H_{\rm ho}}{\hbar\omega} - \frac{3}{2} \quad = \quad b_x^\dagger b_x + b_y^\dagger b_y + b_z^\dagger b_z \equiv N,$$

$$
\begin{aligned}
L_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) &= -i\hbar\left(b_y^\dagger b_z - b_z^\dagger b_y\right), \\
L_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) &= -i\hbar\left(b_z^\dagger b_x - b_x^\dagger b_z\right), \\
L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right) &= -i\hbar\left(b_x^\dagger b_y - b_y^\dagger b_x\right), \\
Q_0 &= \hbar\left(2b_z^\dagger b_z - b_x^\dagger b_x - b_y^\dagger b_y\right), \\
Q_{\pm 1} &= \hbar\sqrt{\frac{3}{2}}\left(\mp b_z^\dagger b_x \mp b_x^\dagger b_z - ib_y^\dagger b_z - ib_z^\dagger b_y\right), \\
Q_{\pm 2} &= \hbar\sqrt{\frac{3}{2}}\left(b_x^\dagger b_x - b_y^\dagger b_y \mp ib_x^\dagger b_y \mp ib_y^\dagger b_x\right), \qquad (16)
\end{aligned}
$$

The first of these is the number operator N which is, up to a constant and a scaling factor, equivalent to the hamiltonian $H_{\rm ho}$. The next three operators are the components of the angular momentum $\bar{L}$ and together they generate the Lie algebra SO(3). This shows that the angular momentum algebra SO(3) is a subalgebra of U(3). The last set of operators in Eq. (16) comprise the five components of the quadrupole operator $\bar{Q}$. Since $\bar{L}$ as well as $\bar{Q}$ conserves the number of bosons, there components commute with the number operator N. This, in fact, is equivalent to the mathematical statement that the algebra U(3) can be written as $\mathrm{U}(1)\otimes\mathrm{SU}(3)$ with $\mathrm{U}(1) = \{N\}$ and $\mathrm{SU}(3) = \{\bar{L}, \bar{Q}\}$.

The above derivation establishes the symmetry algebra of the harmonic oscillator as U(3), which explains the additional degeneracy in its energy spectrum. It should be recognized, however, that these results apply to the Schrödinger equation for a *single particle* and that they can be obtained easily with traditional methods as well. The real power of algebraic methods transpires when one considers the corresponding *many-particle* problem. Raising and lowering operators can be defined for each of the A particles in the system,

$$
\begin{aligned}
b_{x,k}^\dagger &= \frac{1}{\sqrt{2}}\left(x_k' - \frac{\partial}{\partial x_k'}\right), \quad b_{y,k}^\dagger = \frac{1}{\sqrt{2}}\left(y_k' - \frac{\partial}{\partial y_k'}\right), \quad b_{z,k}^\dagger = \frac{1}{\sqrt{2}}\left(z_k' - \frac{\partial}{\partial z_k'}\right), \\
b_{x,k} &= \frac{1}{\sqrt{2}}\left(x_k' + \frac{\partial}{\partial x_k'}\right), \quad b_{y,k} = \frac{1}{\sqrt{2}}\left(y_k' + \frac{\partial}{\partial y_k'}\right), \quad b_{z,k} = \frac{1}{\sqrt{2}}\left(z_k' + \frac{\partial}{\partial z_k'}\right),
\end{aligned}
$$

and by summing over all particles a *total* U(3) algebra can be generated by the nine operators

$$
\sum_{k=1}^{A} b_{i,k}^\dagger b_{j,k}, \qquad \forall i,j \in x,y,z. \qquad (17)
$$

A many-body version of the harmonic-oscillator hamiltonian is of the form

$$
H_{\rm ho} = \hbar\omega \sum_{k=1}^{A} \sum_{i=x,y,z} b_{i,k}^\dagger b_{i,k} + \sum_{k<l} V(k,l). \qquad (18)
$$

The first term on the right-hand side is, up to a constant associated with zero-point motion, identical to the single-particle hamiltonian (13), summed over the A particles in the nucleus. The second term represents a two-body interaction; higher-order interactions can be considered in a similar manner if needed. As shown above, the one-body term commutes with the generators of the total U(3) algebra. The question of genuine interest here is whether there exist classes of two-body interactions that also have the property of commuting with the generators of U(3). This indeed turns out to be the case. The property is related to the structure of the Casimir operators of the (total) algebras SO(3) and SU(3), which can be written as

$$\begin{aligned} C_2[\mathrm{SO}(3)] &= \sum_{k,l} \bar{L}(k)\cdot\bar{L}(l), \\ C_2[\mathrm{SU}(3)] &= \sum_{k,l}\left(\frac{1}{2}\bar{L}(k)\cdot\bar{L}(l)+\frac{1}{6}\bar{Q}(k)\cdot\bar{Q}(l)\right). \end{aligned} \tag{19}$$

An arbitrary combination of a rotational term $\bar{L}\cdot\bar{L}$ and a quadrupole term $\bar{Q}\cdot\bar{Q}$ can thus be written as a combination of the quadratic Casimir operators of SO(3) and SU(3).

This is the basic idea of the SU(3) model which was proposed by Elliott in 1958 to account for rotational phenomena in the spherical shell model [7]. The application of SU(3) to nuclei is more complicated than what is presented in this section since the nucleons have spin and isospin (*i.e.*, there are neutrons and protons) and these have been ignored here. It is an—historically perhaps the first—example of the mechanism of dynamical symmetry breaking, a general discussion of which is presented in the next section.

4. DYNAMICAL SYMMETRY BREAKING

The concept of a dynamical symmetry for which (at least) two algebras G_1 and G_2 with $\mathrm{G}_1 \supset \mathrm{G}_2$ are needed can now be introduced. The eigenstates of a hamiltonian H with symmetry G_1 are labeled as $|\Gamma_1\gamma_1\rangle$. But, since $\mathrm{G}_1 \supset \mathrm{G}_2$, a hamiltonian with G_1 symmetry necessarily must also have a symmetry G_2 and, consequently, its eigenstates can also be labeled as $|\Gamma_2\gamma_2\rangle$. Combination of the two properties leads to the eigenequation

$$H|\Gamma_1\eta_{12}\Gamma_2\gamma_2\rangle = E(\Gamma_1)|\Gamma_1\eta_{12}\Gamma_2\gamma_2\rangle, \tag{20}$$

where the role of γ_1 is played by $\eta_{12}\Gamma_2\gamma_2$. The irreducible representation Γ_2 may occur more than once in Γ_1, and hence an additional quantum number η_{12} is needed to uniquely label the states. Because of G_1 symmetry, eigenvalues of H depend on Γ_1 only.

In many examples in physics (several are discussed below), the condition of G_1 symmetry is too strong and a *possible* breaking of the G_1 symmetry can be imposed via the hamiltonian

$$H' = \kappa_1 C_{m_1}[\mathrm{G}_1] + \kappa_2 C_{m_2}[\mathrm{G}_2], \tag{21}$$

which consists of a combination of Casimir operators of G_1 *and* G_2. The symmetry properties of the hamiltonian H' are now as follows. Since $[H', g_k] = 0$ for all g_k in G_2, H' is invariant under G_2. The hamiltonian H', since it contains $C_{m_2}[\mathrm{G}_2]$, does not

commute, in general, with all elements of G_1 and for this reason the G_1 symmetry is broken. Nevertheless, because H' is a combination of Casimir operators of G_1 and G_2, its eigenvalues can be obtained in closed form,

$$H'|\Gamma_1 \eta_{12} \Gamma_2 \gamma_2\rangle = [\kappa_1 E_{m_1}(\Gamma_1) + \kappa_2 E_{m_2}(\Gamma_2)]\,|\Gamma_1 \eta_{12} \Gamma_2 \gamma_2\rangle. \tag{22}$$

The conclusion is thus that, although H' is not invariant under G_1, its eigenstates are the same as those of H in (20). The hamiltonian H' is said to have G_1 as a dynamical symmetry. The essential feature is that, although the eigenvalues of H' depend on Γ_1 *and* Γ_2 (and hence G_1 is not a symmetry), the eigenstates do not change during the breaking of the G_1 symmetry. As the generators of G_2 are a subset of those of G_1, the dynamical symmetry breaking splits but does not admix the eigenstates. A convenient way of summarizing the symmetry character of H' and the ensuing classification of its eigenstates is as follows:

$$\begin{array}{ccc} G_1 & \supset & G_2 \\ \downarrow & & \downarrow \\ \Gamma_1 & & \eta_{12}\Gamma_2 \end{array} . \tag{23}$$

This equation indicates the larger algebra G_1 (sometimes referred to as the dynamical algebra or spectrum generating algebra) and the symmetry algebra G_2, together with their associated labels with possible multiplicities.

5. THE NUCLEAR SHELL MODEL

The structure of the atomic nucleus is determined, in first approximation, by the nuclear mean field, the average potential felt by a nucleon through the interactions exerted by all others. This average potential is responsible for the shell structure of the nucleus because the energy spectrum of a particle moving in this mean field shows regions with many levels and others with few. A second important ingredient that determines the structure of nuclei is the Pauli principle. Consequently, the nucleus can be viewed as an onion-like construction, with shells determined by the mean-field potential that are being filled in accordance with the Pauli principle. For a description that goes beyond this most basic level, the residual interaction between nucleons must be taken into account and what usually matters most for nuclear structure at low energies is the residual interaction between nucleons in the valence or outer shell. This interaction depends in a complex fashion on the numbers of valence neutrons and protons, and on the valence orbits available to them.

No review is given here of the nuclear shell model which has been the subject of several comprehensive monographs [17, 18, 19, 20, 21]. Instead, after an introductory subsection, describing the model's essential features and assumptions, emphasis is laid on its symmetry structure. It turns out that the two most important correlations in nuclei, pairing and quadrupole, can be analyzed with symmetry techniques.

5.1. The independent-particle shell model

In a non-relativistic approximation, the wave function of any quantum-mechanical state of a nucleus with A nucleons satisfies the Schrödinger equation

$$H\Psi(\xi_1,\xi_2,\ldots,\xi_A) = E\Psi(\xi_1,\xi_2,\ldots,\xi_A), \tag{24}$$

with the hamiltonian

$$H = \sum_{k=1}^{A} \frac{p_k^2}{2m_k} + \sum_{k<l}^{A} W_2(\xi_k,\xi_l) + \sum_{k<l<m}^{A} W_3(\xi_k,\xi_l,\xi_m) + \cdots. \tag{25}$$

The notation ξ_k is used to denote all coordinates of nucleon k, not only its position vector $\bar{r}_k$ but also its spin $\bar{s}_k$ and its isospin $\bar{t}_k$, $\xi_k \equiv \{\bar{r}_k,\bar{s}_k,\bar{t}_k\}$. The term $p_k^2/2m_k$ is the kinetic energy of nucleon k and acts on a single nucleon only. The operator $W_i(\xi_k,\xi_l,\xi_m,\ldots)$ is an i-body interaction between the nucleons $k,l,m,\ldots$, and, as such, acts on i nucleons simultaneously. Since neutron and proton are not elementary particles, it is not *a priori* clear that the interaction should be of two-body nature. Nevertheless, for a presentation of the elementary nuclear shell model, it can be assumed that the nature between the nucleons is two-body, $W_{i>2} = 0$, as will be done in the subsequent discussion.

Under the assumption of at most two-body interactions, one can rewrite (25) as

$$H = \sum_{k=1}^{A} \left(\frac{p_k^2}{2m_k} + V(\xi_k) \right) + \left(\sum_{k<l}^{A} W_2(\xi_k,\xi_l) - \sum_{k=1}^{A} V(\xi_k) \right). \tag{26}$$

The idea is now to choose $V(\xi_k)$ such that the effect of the residual interaction, that is, the second term in (26), is minimized. The independent-particle shell model is obtained by neglecting the residual interaction altogether,

$$H_{\mathrm{ip}} = \sum_{k=1}^{A} \left(\frac{p_k^2}{2m_{\mathrm{n}}} + V(\xi_k) \right), \tag{27}$$

where it is also assumed that all nucleons have the same mass m_{n}. The physical interpretation of the approximation (27) is that each nucleon moves independently in a mean-field potential $V(\xi)$ which represents the average interaction with all other nucleons in the nucleus.

The eigenproblem associated with the hamiltonian (27) is much easier to solve than the original problem (24) because it can be reduced to a one-particle eigenequation. Its solution proceeds as follows. First, one solves the Schrödinger equation of a particle in a potential $V(\xi)$, that is, one finds the eigenfunctions $\phi_i(\xi)$ satisfying

$$\left(\frac{p^2}{2m_{\mathrm{n}}} + V(\xi) \right) \phi_i(\xi) = E_i \phi_i(\xi), \tag{28}$$

where i labels the different eigensolutions. For example, for a harmonic-oscillator potential, the index i stands for the set of quantum numbers n, ℓ and m_ℓ in the eigenfunctions (7). The exact form of the eigenfunctions $\phi_i(\xi)$ depends on the potential $V(\xi)$. For

simple potentials (*e.g.*, the harmonic oscillator) the eigenfunctions can be found in analytic form in terms of standard mathematical functions; for more complicated potentials (*e.g.*, Woods–Saxon) $\phi_i(\xi)$ must be determined numerically. For all 'reasonable' potentials $V(\xi)$ the solutions of (28) can be obtained, albeit in most cases only in numerical form.

The solution of the *many-body* hamiltonian H_{ip} is immediately obtained due to its separability,

$$\Phi_{i_1 i_2 \ldots i_A}(\xi_1, \xi_2, \ldots, \xi_A) = \prod_{k=1}^{A} \phi_{i_k}(\xi_k). \tag{29}$$

Although this is a genuine, mathematical eigensolution of the hamiltonian (27), it is not antisymmetric under the exchange of particles as is required by the Pauli principle. The solution (29) must thus be antisymmetrized which leads to the replacement of the wave function $\Phi_{i_1 i_2 \ldots i_A}(\xi_1, \xi_2, \ldots, \xi_A)$ by a Slater determinant of the form

$$\Psi_{i_1 i_2 \ldots i_A}(\xi_1, \xi_2, \ldots, \xi_A) = \frac{1}{\sqrt{A!}} \begin{vmatrix} \phi_{i_1}(\xi_1) & \phi_{i_1}(\xi_2) & \cdots & \phi_{i_1}(\xi_A) \\ \phi_{i_2}(\xi_1) & \phi_{i_2}(\xi_2) & \cdots & \phi_{i_2}(\xi_A) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{i_A}(\xi_1) & \phi_{i_A}(\xi_2) & \cdots & \phi_{i_A}(\xi_A) \end{vmatrix}. \tag{30}$$

This is the solution of the Schrödinger equation associated with the hamiltonian (27) that takes account of the Pauli principle.

The following question now arises. How should one choose the potential $V(\xi)$ introduced in (26)? This choice can be made at several levels of refinement. Ideally one wants to minimize the expectation value of H in the ground state, that is, to solve the variational equation

$$\delta \int \Psi^*(\xi_1, \xi_2, \ldots, \xi_A) H \Psi(\xi_1, \xi_2, \ldots, \xi_A) d\xi_1 d\xi_2 \ldots d\xi_A = 0. \tag{31}$$

If, in this variational approach, the wave function $\Psi(\xi_1, \xi_2, \ldots, \xi_A)$ is allowed to vary freely, the solution of (31) is equivalent to the ground-state solution of the Schrödinger equation (24). Obviously, one needs to set more modest goals to arrive at a solvable problem! One way to do so is to restrict $\Psi(\xi_1, \xi_2, \ldots, \xi_A)$ in (31) to the form of a Slater determinant, in other words, to minimize the ground-state energy by varying the potential $V(\xi)$ that defines the single-particle wave functions $\phi_{i_1}, \phi_{i_2}, \ldots, \phi_{i_A}$ in (30). This is known as the Hartree–Fock method. One determines the form of the potential $V(\xi)$ by requiring the expectation value of the *complete* hamiltonian (25) in the state (30) to be minimal.

The ground-state energy determined in Hartree–Fock theory is not the correct one; nevertheless, it is the best procedure at hand to construct an independent-particle model. Often an even simpler approach is preferred. One proposes a phenomenological form of the potential $V(\xi)$, such that the Schrödinger equation associated with H_{ip} in (27) is analytically solvable. The potential which best mimics the nuclear mean-field potential and which can be solved exactly, is the harmonic-oscillator potential discussed in Section 3.

An important quantity appearing in the harmonic-oscillator model is the elementary quantum of excitation $\hbar\omega$. By relating the radius of the nucleus, R, to the number of

nucleons, A, and subsequently deriving a relationship between R, A and the oscillator length parameter b, one finds the expression [18]

$$b \approx 1.00A^{1/6}\ \text{fm}, \tag{32}$$

and, since $b = \sqrt{\hbar/m_{\text{n}}\omega}$,

$$\hbar\omega \approx 41A^{-1/3}\ \text{MeV}. \tag{33}$$

The harmonic-oscillator solutions $\phi_{n\ell m_\ell}(r,\theta,\varphi)$ of Eq. (7) contain the dependence on the spatial coordinates only and not on the intrinsic spin of the particle. Since the intrinsic spin does not appear in the potential (6), the wave functions are simply given by the product

$$\phi_{n\ell m_\ell}(r,\theta,\varphi)\chi_{sm_s}, \tag{34}$$

where χ_{sm_s} are spinors for particles with intrinsic spin $s = \frac{1}{2}$. The energies are independent of m_s and are still given by Eq. (8). The eigenstates (34) do not have good *total* angular momentum, that is, they are not eigenstates of j^2 where $\bar{j}$ results from the coupling of the orbital angular momentum $\bar{\ell}$ and the spin $\bar{s}$ of the nucleon. States of good angular momentum are constructed from (34) with the help of Clebsch–Gordan coefficients,

$$\phi_{n\ell jm_j}(r,\theta,\varphi) = \sum_{m_\ell m_s} (\ell m_\ell\, sm_s|jm_j)\phi_{n\ell m_\ell}(r,\theta,\varphi)\chi_{sm_s}. \tag{35}$$

Again, this state has the same energy eigenvalue (8) since all states appearing in the sum are degenerate. If the spin degeneracy of the quantum numbers $(n\ell jm_j)$ is taken into account, stable shell gaps are obtained at the nucleon numbers 2, 8, 20, 40, 70, 112,.... These are the magic numbers of the harmonic oscillator in three dimensions.

The existence of nuclear shell structure can be demonstrated in a variety of ways. The most direct way is by measuring the ease with which a nucleus can be excited. If it has a closed shell structure, one expects it to be rather stable and difficult to excite. This should be particularly so for nuclei that are doubly magic, that is, nuclei with a closed-shell configuration for neutrons *and* protons. The principle is illustrated in Fig. 2. The figure shows the energy $E_{\text{x}}(2_1^+)$ of the first-excited 2^+ state relative to the ground state for all even–even nuclei. This energy is multiplied with $A^{1/3}$ and the result plotted on a normalized scale. (The factor $A^{1/3}$ accounts for the gradual decrease with mass number A of the strength of the nuclear residual interaction which leads a compression of the spectrum with A.) Nuclei with particularly high values of $E_{\text{x}}(2_1^+)A^{1/3}$ are ^{16}O ($N = Z = 8$), ^{40}Ca ($N = Z = 20$), ^{48}Ca ($N = 28$, $Z = 20$), ^{132}Sn ($N = 82$, $Z = 50$) and ^{208}Pb ($N = 126$, $Z = 82$). Figure 2 establishes the stability properties of the isotopes and/or isotones with $N, Z = 8$, 20, 28, 50, 82 and 126.

How to explain the differences between the observed magic numbers (2, 8, 20, 28, 50, 82 and 126) and those of the harmonic oscillator? The observed ones can be reproduced in an independent-particle model if to the harmonic-oscillator hamiltonian H_{ho} a spin–orbit as well as an orbit–orbit term is added of the form

$$V_{\text{so}} = \zeta_{\text{so}}(r)\bar{\ell}\cdot\bar{s}, \qquad V_{\text{oo}} = \zeta_{\text{oo}}(r)\bar{\ell}\cdot\bar{\ell}. \tag{36}$$

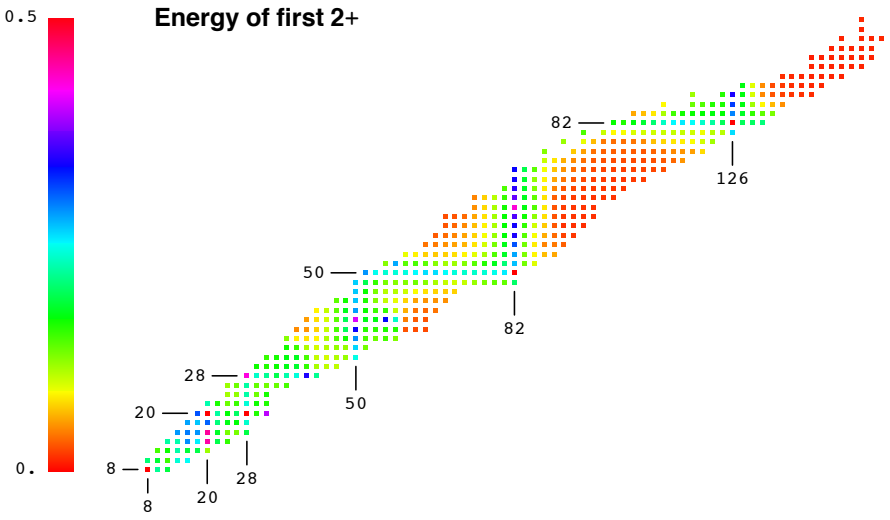

FIGURE 2. The energy of the first-excited 2^+ state in all even–even nuclei with $N, Z \geq 8$ (where known experimentally) plotted as a function of neutron number N along the x axis and proton number Z along the y axis. The excitation energy is multiplied by $A^{1/3}$ and subsequently normalized to 1 for ^{208}Pb where this quantity is highest. The value of $E_x(2_1^+)A^{1/3}$ is indicated by the scale shown on the left. To improve the resolution of the plot, the scale only covers part of the range from 0 to 0.5 since only a few doubly magic nuclei (^{16}O, ^{40,48}Ca, ^{132}Sn and ^{208}Pb) have values greater than 0.5.

The eigenvalue problem associated with the hamiltonian $H_{\rm ho} + V_{\rm so} + V_{\rm oo}$ is not, in general, analytically solvable but the dominant characteristics can be found from the expectation values

$$\langle n\ell j m_j | V_{\rm so} | n\ell j m_j \rangle = \tfrac{1}{2} \langle \zeta_{\rm so}(r) \rangle_{n\ell} \left[j(j+1) - \ell(\ell+1) - \tfrac{3}{4} \right], \tag{37}$$

and

$$\langle n\ell j m_j | V_{\rm oo} | n\ell j m_j \rangle = \ell(\ell+1) \langle \zeta_{\rm oo}(r) \rangle_{n\ell}, \tag{38}$$

with radial integrals defined as

$$\langle \zeta(r) \rangle_{n\ell} = \int_0^{+\infty} \zeta(r) R_{n\ell}(r) R_{n\ell}(r) r^2 \, dr. \tag{39}$$

Consequently, the degeneracy of the single-particle levels within one major oscillator shell is lifted. Empirically, one finds that the radial integrals approximately satisfy the relations [18]

$$\langle \zeta_{\rm so}(r) \rangle_{n\ell} \approx -20 A^{-2/3} \text{ MeV}, \qquad \langle \zeta_{\rm oo}(r) \rangle_{n\ell} \approx -0.1 \text{ MeV}. \tag{40}$$

The origin of the orbit–orbit coupling can be understood from elementary arguments. The corrections to the harmonic-oscillator potential are repulsive for short and large distances and attractive for intermediate distances. These corrections therefore favour large-ℓ over small-ℓ orbits. The spin–orbit coupling has a relativistic origin. An important feature is that the radial integral is negative, reflecting the empirical finding that states with parallel spin and orbital angular momentum are pushed down in energy while in the antiparallel case they are pushed up.

5.2. A symmetry triangle for the shell model

The summary of the preceding discussion is that a simple approximation of the nuclear mean-field potential consists of a three-dimensional harmonic oscillator corrected with spin–orbit and orbit–orbit terms. If, in addition, a two-body residual interaction is included, the many-body hamiltonian acquires the following form:

$$H = \sum_{k=1}^{A} \left(\frac{p_k^2}{2m_{\rm n}} + \frac{1}{2} m_{\rm n} \omega^2 r_k^2 + \zeta_{\rm oo}\, \bar{\ell}_k \cdot \bar{\ell}_k + \zeta_{\rm so}\, \bar{\ell}_k \cdot \bar{s}_k \right) + \sum_{k<l} V_{\rm res}(\xi_k, \xi_l), \qquad (41)$$

where the indices in the second sum run over a *restricted* number of particles, usually only the valence nucleons. In spite of the severe simplifications of the original many-body problem (24), the solution of the Schrödinger equation associated with the hamiltonian (41) still represents a formidable problem since the residual interaction must be diagonalized in a basis of Slater determinants of the type (30). Even if one limits oneself to valence excitations, the dimension of the Hilbert space rapidly explodes with increasing mass of the nucleus. The *m*-scheme basis can be used to illustrate this. Because of the antisymmetry of Slater determinants, their number can be computed easily. For n valence neutrons and z valence protons distributed over Ω_n and Ω_z single-particle states in the valence shell, the dimension of the basis is

$$\frac{\Omega_n!}{n!(\Omega_n - n)!} \frac{\Omega_z!}{z!(\Omega_z - z)!}. \qquad (42)$$

Application of this formula to ^{28}Si (in the *sd* shell, $\Omega_n = \Omega_z = 12, n = z = 6$) and to ^{78}Y (half-way between the magic numbers 28 and 50, $\Omega_n = \Omega_z = 22, n = z = 11$) illustrates the point since it leads to dimensions of 8.5 10^5 and 5.0 10^{11}, respectively.

Given the considerable effort it takes to solve the nuclear many-body problem even only approximately, any analytical solution of (41) that can be obtained through symmetry techniques might be of considerable value. In fact, the residual nuclear interaction can approximately be written as pairing-plus-quadrupole,

$$V_{\rm res}(\xi_k, \xi_l) = V_{\rm pairing}(\bar{r}_k, \bar{r}_l) + V_{\rm quadrupole}(\bar{r}_k, \bar{r}_l), \qquad (43)$$

where the exact form of these interactions is defined below. For particular values of the parameters in the mean field and if the residual interaction is either of pairing or of quadrupole type, the eigenproblem (41) can be solved analytically. Three situations arise, of which two are of interest:

1. *No residual interaction.* If $V_{\rm res}(\xi_k, \xi_l) = 0$, the solution of (41) reduces to a Slater determinant built from harmonic-oscillator eigenstates.
2. *Pairing interaction.* If the residual interaction has a pure pairing character, Racah's SU(2) model of pairing results. This model is usually applied in the jj-coupling limit of strong spin–orbit coupling.
3. *Quadrupole interaction.* If the residual interaction has a pure quadrupole character, Elliott's SU(3) model of rotation results. This model requires an *LS*-coupling scheme which occurs in the absence of spin–orbit coupling.

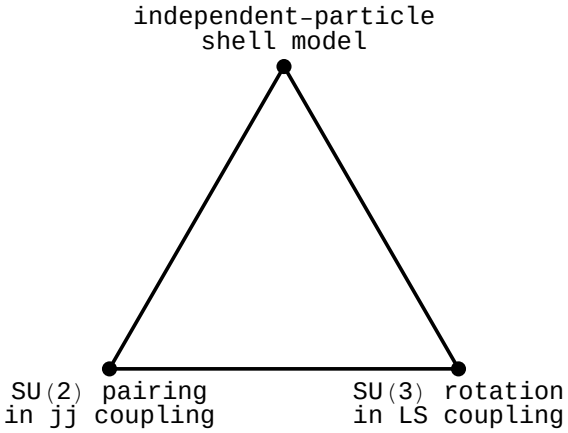

FIGURE 3. Schematic representation of the shell-model parameter space with its three analytically solvable vertices.

The situation is represented schematically in Fig. 3. It should be emphasized that, in contrast to the top vertex, the two bottom vertices, SU(2) and SU(3), represent solutions of the nuclear hamiltonian which include genuine many-body correlations. These two limits are thus of particular interest. Furthermore, it should be stressed that the triangle in Fig. 3 is a highly schematic representation of a complex parameter space, and that SU(2) and SU(3) are the most elementary examples of symmetries that can be encountered in the shell model. The two basic symmetries and their generalizations are discussed in the following two subsections, which draw on material from earlier reviews [22, 23].

5.3. Pairing models

The discussion of pairing correlations in nuclei traditionally has been inspired by the treatment of superfluidity in condensed matter. The superfluid phase in the latter systems is characterized by the presence of a large number of identical bosons in a single quantum state. In superconductors the bosons are pairs of electrons with opposite momenta that form at the Fermi surface. In nuclei the bosons are pairs of valence nucleons with opposite angular momenta.

Condensed-matter superfluidity (and associated superconductivity) was explained by Bardeen, Cooper and Schrieffer [24] and the resulting BCS theory has strongly influenced the discussion of pairing in nuclei [25]. Nevertheless, the approximations made in BCS theory are less appropriate for nuclei since the number of nucleons is comparatively small, and over the years techniques have been developed to yield exact solutions of the nuclear pairing problem. These methods, often based on group theory, are summarized in the present subsection.

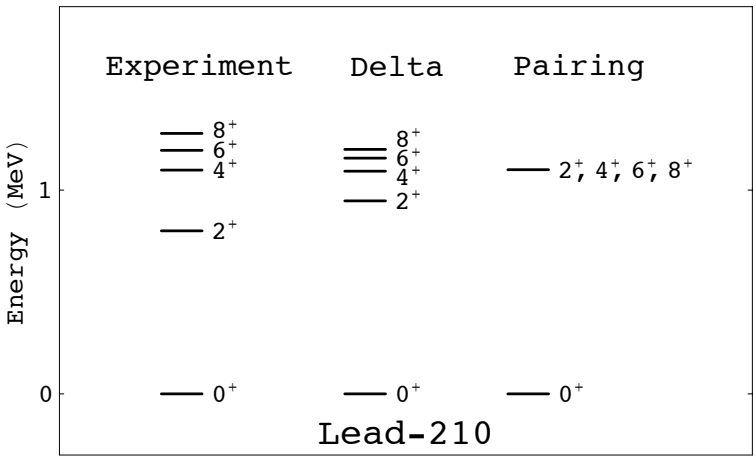

FIGURE 4. Observed [26] energy spectrum of ^{210}Pb (left), and the corresponding spectra for a delta (middle) and for a pairing interaction (right). Levels are labelled by their angular momentum and parity J^{π}.

5.3.1. Pairing, seniority and quasi-spin algebras

The pairing interaction is a reasonable first-order approximation to the strong force between identical nucleons. For nucleons in a single-j shell, pairing is defined by the matrix elements

$$v_J \equiv \langle j^2; JM_J | V_{\text{pairing}} | j^2; JM_J \rangle = -g_0(2j+1)\delta_{J0}, \tag{44}$$

where j is the total (orbital+spin) angular momentum of a single nucleon (hence j is half-odd-integer), J results from the coupling of two js and M_J is the projection of J on the z axis. Furthermore, g_0 is the strength of the interaction which is attractive in nuclei ($g_0 > 0$). The pairing interaction is illustrated in Fig. 4 for the nucleus ^{210}Pb which can be described as two neutrons in a $2g_{9/2}$ orbit outside the doubly magic ^{208}Pb inert core. For two non-interacting valence neutrons all levels would be degenerate in energy. This degeneracy is lifted by the residual interaction which has a short-range attractive character. The spectrum obtained with a delta interaction is close to what is observed experimentally, and a pairing interaction constitutes a further approximation.

The pairing interaction was introduced by Racah for the classification of n electrons in an atom [27]. He was able to derive a closed formula for the interaction energy among the n electrons and to prove that any eigenstate of the pairing interaction is characterized by a 'seniority number' υ which corresponds to the number of electrons that are not in pairs coupled to orbital angular momentum $L = 0$. Racah's original definition of seniority made use of coefficients of fractional parentage. He later noted that simplifications arose through the use of group theory [28]. Seniority turned out to be a label associated with the (unitary) symplectic algebra $\mathrm{Sp}(2j+1)$ in the classification

$$\begin{array}{ccccc} \mathrm{U}(2j+1) & \supset & \mathrm{Sp}(2j+1) & \supset & \mathrm{SU}(2) \\ \downarrow & & \downarrow & & \downarrow \\ [1^n] & & [1^{\upsilon}] & & J \end{array} . \tag{45}$$

If the nucleons are identical, all states of the j^n configuration belong to the totally antisymmetric irreducible representation (IR) $[1^n]$ of U$(2j+1)$. The IRs of Sp$(2j+1)$ therefore must also be totally antisymmetric of the type $[1^\upsilon]$ with allowed values of seniority $\upsilon = n, n-2, \ldots, 1$ or 0. The angular momentum content for a given seniority υ can also be worked out [29] but no simple general rule is available for the reduction Sp$(2j+1) \supset$ SU(2).

An alternative, simpler definition of seniority can be given which relies on the existence of an SU(2) symmetry of the pairing hamiltonian [30, 31]. In second quantization the pairing interaction (44) is written as

$$V_{\text{pairing}} = -g_0 S^j_+ S^j_-, \tag{46}$$

with

$$S^j_+ = \frac{1}{2}\sqrt{2j+1}\,(a^\dagger_j \times a^\dagger_j)^{(0)}_0, \qquad S^j_- = \left(S^j_+\right)^\dagger, \tag{47}$$

where $a^\dagger_{jm_j}$ creates a nucleon in the shell j with projection m_j. The commutator of S^j_+ and S^j_- leads to the operator $[S^j_+, S^j_-] = (2n_j - 2j - 1)/2 \equiv 2S^j_z$, which thus equals, up to a constant, the number operator n_j. Since the three operators $\{S^j_z, S^j_\pm\}$ close under commutation, $[S^j_z, S^j_\pm] = \pm S^j_\pm$ and $[S^j_+, S^j_-] = 2S^j_z$, they form an SU(2) algebra, referred to as the quasi-spin algebra.

This algebraic structure allows an analytical solution of the pairing hamiltonian. From the commutation relations it follows that $S^j_+ S^j_- = (S^j)^2 - (S^j_z)^2 + S^j_z$, which shows that the pairing hamiltonian can be written as a combination of Casimir operators belonging to SU(2) and SO$(2) \equiv \{S^j_z\}$. The associated eigenvalue problem can be solved instantly, yielding the energy expression $-g_0[S(S+1) - M_S(M_S-1)]$. The quantum numbers S and M_S can be put in relation to the seniority υ and the nucleon number n, $S = (2j - 2\upsilon + 1)/4$ and $M_S = (2n - 2j - 1)/4$, leading to the energy expression $-g_0(n-\upsilon)(2j - n - \upsilon + 3)/4$. This coincides with the original expression given by Racah, Eq. (50) of Ref. [27], after the replacement of the degeneracy in *LS* coupling, $4\ell + 2$, by the degeneracy in jj coupling, $2j+1$.

5.3.2. Seniority in several j shells

The quasi-spin algebra can be generalized to the case of several (say s) degenerate shells by making the substitutions $S^j_+ \mapsto S_+ \equiv \sum_j S^j_+$ and $2j+1 \mapsto \sum_j (2j+1) \equiv 2\Omega$. Therefore, if a semi-magic nucleus can be approximated as a system of identical nucleons interacting through a pairing force and distributed over several degenerate shells, the formulas of the quasi-spin formalism should apply. In particular, the ground states of even–even semi-magic nuclei will have a 'superfluid' structure of the form

$$(S_+)^{n/2}\,|\mathrm{o}\rangle, \tag{48}$$

where $|\mathrm{o}\rangle$ represents the vacuum (*i.e.*, the doubly-magic core nucleus).

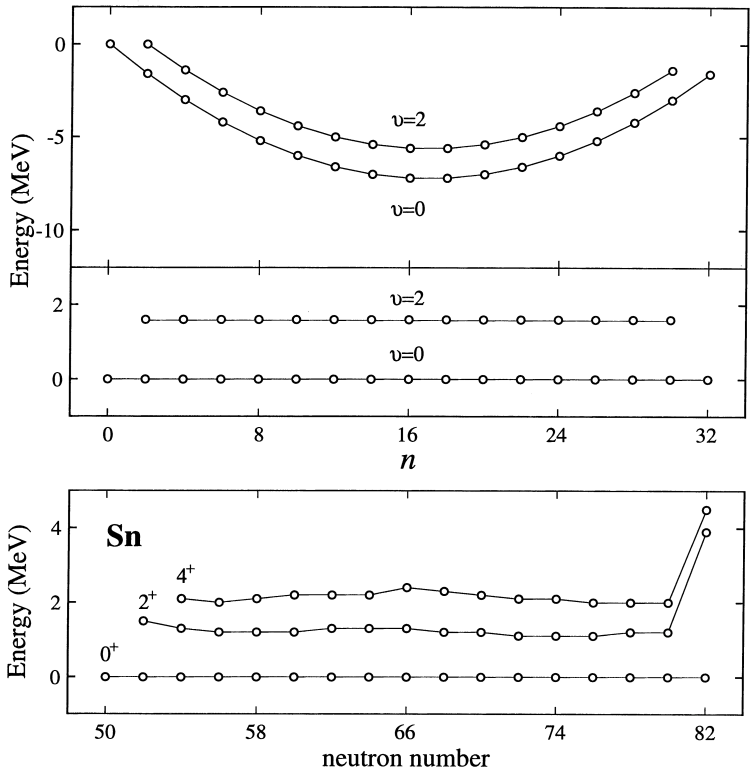

FIGURE 5. The difference $E(n,2)-E(n,0)$ is a function of particle number n (top) and the corresponding observed excitation energies $E_{\rm x}(2_1^+)\equiv E(2_1^+)-E(0_1^+)$ and $E_{\rm x}(4_1^+)\equiv E(4_1^+)-E(0_1^+)$ in the Sn isotopes.

The SU(2) quasi-spin solution of the pairing hamiltonian (46) leads to several characteristic predictions: a constant excitation energy (independent of n) of the first-excited 2^+ state in even–even isotopes, the linear variation of two-nucleon separation energies as a function of n, the odd–even staggering in nuclear binding energies, the enhancement of two-nucleon transfer. The first of these predictions is illustrated in Fig. 5. The ground state of an even–even nucleus has $\upsilon=0$ and the lowest excited states have $\upsilon=2$. An example of such $\upsilon=2$ states are those in a two-nucleon j^2 configuration with $J\neq 0$, $J=2,4,\ldots,2j-1$. The energy difference between $\upsilon=2$ and $\upsilon=0$ states is given by

$$E(n,2)-E(n,0)=g_0\Omega, \tag{49}$$

and is independent of the number of valence nucleons. This prediction is illustrated in Fig. 5 where it is compared with the excitation energies of the 2_1^+ and 4_1^+ levels in the even–even Sn isotopes.

Another manifestation of pairing correlations can be obtained from two-nucleon separation energies defined as

$$S_{2\rm n}(N,Z)=B(N,Z)-B(N-2,Z),\qquad S_{2\rm p}(N,Z)=B(N,Z)-B(N,Z-2), \tag{50}$$

for two-neutron and two-proton separation energies, respectively, where $B(N,Z)$ denotes the ground-state binding energy of a nucleus with N neutrons and Z protons. In a simple

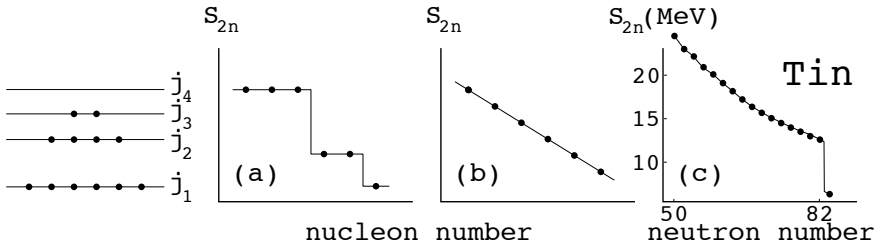

FIGURE 6. The two-nucleon separation energy S_{2n} or S_{2p} as an indicator of pairing. If there are no pairing correlations among the nucleons occupying the levels shown on the left, the separation energy, as a function of nucleon number, behaves as in (a). Superfluidity leads to the behaviour shown in (b). The observed [32] two-neutron separation energies in (c) show that the superfluid solution is appropriate for the Sn isotopes with active neutrons in the 50–82 shell.

approximation the binding energy of the ground state of a semi-magic nucleus can be related to the pairing interaction energy among its valence nucleons [21]. This leads to the following result for the difference of two-nucleon separation energies:

$$S_{2n}(N,Z) - S_{2n}(N-1,Z) = -g_0, \qquad S_{2p}(N,Z) - S_{2p}(N,Z-1) = -g_0, \tag{51}$$

that is, the two-nucleon separation energy varies linearly as a function of nucleon number. For a system of identical nucleons occupying a set of non-degenerate single-particle levels as shown on the left of Fig. 6, the complete absence of pairing correlations ($g_0 = 0$) would lead to a staircase behaviour of S_{2n} (S_{2p}) as a function of N (Z) (see Fig. 6a). The other extreme, strong pairing correlations among nucleons distributed over closely spaced single-particle levels, is represented in Fig. 6b which shows a smooth decrease of S_{2n} or S_{2p} as the nucleon number increases. Figure 6c shows the two-neutron separation energies measured in the Sn isotopes, as a function of neutron number. As far as the 50–82 shell is concerned, the data are consistent with the superfluid solution. At $N = 82$ a large jump in S_{2n} is observed. This indicates that pairing correlations are confined to the 50–82 shell.

Exact quasi-spin SU(2) is valid only for identical nucleons interacting through a pairing force in a single-j shell or distributed over several, degenerate shells. A more generally valid model is obtained if one imposes the following condition on the hamiltonian:

$$[[H,S_+],S_+] = \Delta\,(S_+)^2\,, \tag{52}$$

where S_+ creates the lowest two-nucleon eigenstate of H and Δ is a constant. This condition of generalized seniority, which was proposed by Talmi [33], is much weaker than the assumption of a pairing interaction and, in particular, it does not require the commutator $[S_+,S_-]$ to yield (up to a constant) the number operator—a property which is central to the quasi-spin formalism. In spite of the absence of a closed algebraic structure, it is still possible to compute the exact ground-state eigenvalue but hamiltonians satisfying (52) are no longer necessarily completely solvable.

An exact method to solve the problem of identical nucleons distributed over non-degenerate levels interacting through a pairing force was proposed a long time ago by Richardson [34] based on the Bethe *ansatz* [35]. As an illustration of Richardson's

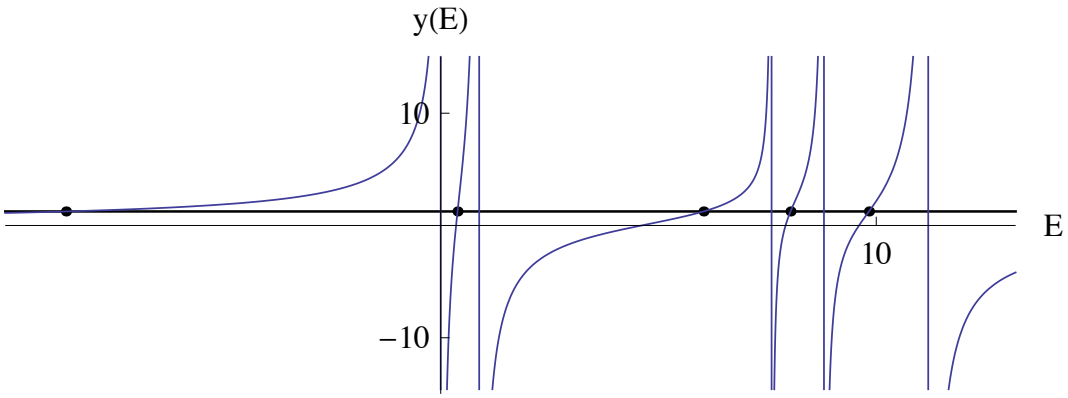

FIGURE 7. Graphical solution of the Richardson equation for $n = 2$ fermions distributed over $s = 5$ single-particle orbits. The sum $\sum_j \Omega_j/(2\varepsilon_j - E) \equiv y(E)$ is plotted as a function of E; the intersections of this curve with the line $y = 1/g_0$ (dots) then correspond to the solutions of the Richardson equation.

approach, consider the pairing interaction supplemented with a one-body term:

$$H = \sum_j \varepsilon_j n_j - g_0 S_+ S_- = \sum_j \varepsilon_j n_j - g_0 \sum_j S_+^j \sum_{j'} S_-^{j'}, \tag{53}$$

where ε_j are single-particle energies. The solvability of the hamiltonian (53) arises as a result of the symmetry $\mathrm{SU}(2) \otimes \mathrm{SU}(2) \otimes \cdots$, where each SU(2) algebra pertains to a specific j. Whether the solution of (53) can be called superfluid depends on the differences $\varepsilon_j - \varepsilon_{j'}$ in relation to the strength g_0. In all cases the solution is known in closed form for all possible choices of ε_j.

It is instructive to analyze first the case of $n = 2$ nucleons because it gives insight into the structure of the general problem. The two-nucleon, $J = 0$ eigenstates can be written as $S_+|\mathrm{o}\rangle = \sum_j x_j S_+^j |\mathrm{o}\rangle$ with x_j coefficients that are determined from the eigenequation $HS_+|\mathrm{o}\rangle = ES_+|\mathrm{o}\rangle$ where E is the unknown eigenenergy. With some elementary manipulations this can be converted into the secular equation $2\varepsilon_j x_j - g_0 \sum_{j'} \Omega_{j'} x_{j'} = E x_j$, with $\Omega_j = j + \frac{1}{2}$, from where x_j can be obtained up to a normalization constant, $x_j \propto g_0/(2\varepsilon_j - E)$. The eigenenergy E can be found by substituting the solution for x_j into the secular equation, leading to

$$\sum_j \frac{\Omega_j}{2\varepsilon_j - E} = \frac{1}{g_0}. \tag{54}$$

This equation can be solved graphically which is done in Fig. 7 for a particular choice of single-particle energies ε_j and degeneracies Ω_j. In the limit $g_0 \to 0$ of weak pairing, the solutions $E \to 2\varepsilon_j$ are obtained, as should be. Of more interest is the limit of strong pairing, $g_0 \to +\infty$. From the graphical solution we see that in this limit there is one eigenstate of the pairing hamiltonian which lies well below the other eigenstates with approximately constant amplitudes x_j since for that eigenstate $|E| \gg 2|\varepsilon_j|$. Hence, in

the limit of strong pairing one finds a $J = 0$ ground state which can be approximated as

$$S_+^{\rm c}|{\rm o}\rangle \approx \sqrt{\frac{1}{\Omega}}\sum_j S_+^j|{\rm o}\rangle, \tag{55}$$

where $\Omega = \sum_j \Omega_j$. Because of this property this state is often referred to as the collective S state, in the sense that all single-particle orbits contribute to its structure.

This result can be generalized to n particles, albeit that the general solution is more complex. On the basis of the two-particle problem one may propose, for an even number of particles n, a ground state of the hamiltonian (53) of the form (up to a normalization constant)

$$\prod_{\alpha=1}^{n/2}\left(\sum_j \frac{1}{2\varepsilon_j - E_\alpha}S_+^j\right)|{\rm o}\rangle, \tag{56}$$

which is known as the Bethe *ansatz* [35]. Each pair in the product is defined through coefficients $x_j = (2\varepsilon_j - E_\alpha)^{-1}$ in terms of an energy E_α depending on α which labels the $n/2$ pairs. This product indeed turns out to be the ground state provided the E_α are solutions of $n/2$ coupled, non-linear equations

$$\sum_j \frac{\Omega_j}{2\varepsilon_j - E_\alpha} - \sum_{\beta(\neq\alpha)}^{n/2}\frac{2}{E_\beta - E_\alpha} = \frac{1}{g_0}, \qquad \alpha = 1,\ldots,n/2, \tag{57}$$

known as the Richardson equations [34]. Note the presence of a second term on the left-hand side with differences of the unknowns $E_\beta - E_\alpha$ in the denominator, which is absent in the two-particle case. In addition, the energy of the state (56) is given by $\sum_\alpha E_\alpha$. A characteristic feature of the Bethe *ansatz* is that it no longer consists of a superposition of *identical* pairs since the coefficients $(2\varepsilon_j - E_\alpha)^{-1}$ vary as α runs from 1 to $n/2$. Richardson's model thus provides a solution that covers all possible hamiltonians (53), ranging from those with superfluid character to those with little or no pairing correlations [36].

An important remaining restriction on the form of the pairing hamiltonian (53) is that it contains a single strength parameter g_0 whereas, in general, the interaction might depend on j and j', leading to $s(s+1)/2$ strengths $g_0(jj') = g_0(j'j)$. In nuclei, often the assumption of a separable interaction is made which, in the case of pairing, leads to strengths $g_0(jj') = g_0 c_j c_{j'}$ in terms of s parameters c_j. This restriction leads to the following pairing hamiltonian:

$$H = \sum_j \varepsilon_j n_j - g_0 \sum_{jj'} c_j c_{j'} S_+^j S_-^{j'}. \tag{58}$$

As yet, no closed solution of the general hamiltonian (58) is known but three solvable cases have been worked out:

1. The strengths c_j are constant (independent of j). This case was discussed above.
2. The single-particle energies ε_j are constant (independent of j). The solution was given by Pan *et al.* [37]
3. There are two levels. The solution was given by Balantekin and Pehlivan [38].

5.3.3. Seniority with neutrons and protons

About ten years after its introduction by Racah, seniority was adopted in nuclear physics for the jj-coupling classification of nucleons in a single-j shell [39, 40]. The main additional difficulty in nuclei is that one deals with a system of neutrons and protons, and hence the isospin T of the nucleons should be taken into account. The generalization of the classification (45) for identical nucleons toward neutrons and protons reads as follows:

$$\begin{array}{ccccccccc} \mathrm{U}(4j+2) & \supset & \Big(\mathrm{U}(2j+1) & \supset & \mathrm{Sp}(2j+1) & \supset & \mathrm{SU}(2)\Big) & \otimes & \mathrm{SU}_T(2) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ [1^n] & & [h] & & [\sigma] & & J & & T \end{array} , \tag{59}$$

where $[h]$ and $[\sigma]$ are Young tableaux associated with $\mathrm{U}(2j+1)$ and $\mathrm{Sp}(2j+1)$. In general, $2j+1$ labels are needed to characterize an IR of $\mathrm{U}(2j+1)$, $[h]=[h_1,h_2,\ldots,h_{2j+1}]$, and $j+\frac{1}{2}$ labels are needed for an IR of $\mathrm{Sp}(2j+1)$, $[\sigma]=[\sigma_1,\sigma_2,\ldots,\sigma_{j+1/2}]$. To ensure overall antisymmetry under $\mathrm{U}(4j+2)$, the Young tableaux of $\mathrm{U}(2j+1)$ and $\mathrm{U}_T(2)$ must be conjugate, that is, one is obtained from the other by interchanging rows and columns. Since the Young tableau associated with $\mathrm{U}_T(2)$ is determined by the nucleon number n and the total isospin T as $[n/2+T,n/2-T]$, the Young tableau of $\mathrm{U}(2j+1)$ must therefore be

$$[h]=[\overbrace{2,2,\ldots,2}^{n/2-T},\overbrace{1,1,\ldots,1}^{2T}]. \tag{60}$$

Since an IR of $\mathrm{U}(2j+1)$ has at most $2j+1$ labels, it follows that $n/2+T\leq 2j+1$. Furthermore, all non-zero labels in $[\sigma]$ must be either 2 or 1 and the Young tableau of $\mathrm{Sp}(2j+1)$ must therefore be of the form

$$[\sigma]=[\overbrace{2,2,\ldots,2}^{\upsilon/2-t},\overbrace{1,1,\ldots,1}^{2t}]. \tag{61}$$

The IR of $\mathrm{Sp}(2j+1)$ is thus characterized by *two* labels [40]: the seniority υ and the 'reduced isospin' t. The former has the same interpretation as in the like-nucleon case while the latter corresponds to the isospin of the nucleons which are not in pairs coupled to $J=0$.

The group-theoretical analysis is considerably more complex here than in the case of identical nucleons and, in addition, for each value of j one is faced with a different reduction problem associated with $\mathrm{U}(2j+1)\supset\mathrm{Sp}(2j+1)\supset\mathrm{SU}(2)$. It is therefore advantageous to go over to a quasi-spin formulation of the problem and, as was shown by Helmers [31], this is possible for whatever value of the intrinsic quantum number of the particles (which is $t=\frac{1}{2}$ for nucleons). If the pairing interaction is assumed to be isospin invariant, it is the same in the three $T=1$ channels, neutron–neutron, neutron–proton and proton–proton, and Eq. (46) can be generalized to

$$V'_{\text{pairing}}=-g_0\sum_{\mu}S_{+,\mu}S_{-,\mu}=-g_0S_+\cdot S_-, \tag{62}$$

where the dot indicates a scalar product in isospin. In terms of the nucleon creation operators $a^\dagger_{jm_jtm_t}$, which now carry also isospin indices (with $t = \frac{1}{2}$), the pair operators are

$$S_{+,\mu} = \frac{1}{2}\sum_j \sqrt{2j+1}(a^\dagger_{jt} \times a^\dagger_{jt})^{(01)}_{0\mu}, \qquad S_{-,\mu} = \left(S_{+,\mu}\right)^\dagger, \tag{63}$$

where the coupling refers to angular momentum and to isospin. The index μ (isospin projection) distinguishes neutron–neutron ($\mu = +1$), neutron–proton ($\mu = 0$) and proton–proton ($\mu = -1$) pairs. There are thus three different pairs with $J = 0$ and $T = 1$ and they are connected by the isospin raising and lowering operators $T_\pm$. By considering the commutation relations between the different operators, a closed algebraic structure is obtained, generated by the pair operators $S_{\pm,\mu}$, the number operator n and the isospin operators $T_\pm$ and T_z. The quasi-spin algebra of neutrons and protons in degenerate j shells turns out to be SO(5), by virtue of which the hamiltonian (62) is analytically solvable [41, 42].

A further generalization is possible in *LS* coupling. For a neutron and a proton there exists a different paired state with *parallel* spins. The most general pairing interaction for a system of neutrons and protons is therefore of the form

$$V''_{\text{pairing}} = -g_0 S_+ \cdot S_- - g'_0 P_+ \cdot P_-, \tag{64}$$

where the pair operators are defined as

$$\begin{aligned} S_{+,\mu} &= \sqrt{\frac{1}{2}}\sum_\ell \sqrt{2\ell+1}(a^\dagger_{\ell st} \times a^\dagger_{\ell st})^{(001)}_{00\mu}, \qquad S_{-,\mu} = \left(S_{+,\mu}\right)^\dagger, \\ P_{+,\mu} &= \sqrt{\frac{1}{2}}\sum_\ell \sqrt{2\ell+1}(a^\dagger_{\ell st} \times a^\dagger_{\ell st})^{(010)}_{0\mu 0}, \qquad P_{-,\mu} = \left(P_{+,\mu}\right)^\dagger, \end{aligned} \tag{65}$$

where $a^\dagger_{\ell m_\ell s m_s t m_t}$ creates a nucleon in the shell ℓ with projection m_ℓ, spin projection m_s and isospin projection m_t. The hamiltonian (64) contains two parameters g_0 and g'_0, the strengths of the isovector and isoscalar components of the pairing interaction. While in the previous case the single strength parameter just defines an overall scale, this is no longer true for a generalized pairing interaction and different solutions are obtained for different ratios g_0/g'_0.

In general, the eigenproblem associated with the interaction (64) can only be solved numerically; for specific choices of g_0 and g'_0 the solution of V''_{pairing} can be obtained analytically [43, 44]. A closed algebraic structure is obtained, formed by the pair operators (65), their commutators, the commutators of these among themselves, and so on until closure is attained. The quasi-spin algebra in this case turns out to be SO(8), with 28 generators, consisting of the pair operators $S_{\pm,\mu}$ and $P_{\pm,\mu}$, the number operator n, the spin and isospin operators S_μ and T_μ, and the Gamow–Teller-like operator $Y_{\mu\nu}$, which is a vector in spin and isospin. The symmetry character of the hamiltonian (64) is obtained by studying the subalgebras of SO(8). Of relevance are the subalgebras $\text{SO}_T(5) \equiv \{S_{\pm,\mu}, n, T_\mu\}$, $\text{SO}_T(3) \equiv \{T_\mu\}$, $\text{SO}_S(5) \equiv \{P_{\pm,\mu}, n, S_\mu\}$, $\text{SO}_S(3) \equiv \{S_\mu\}$ and

SO(6) $\equiv \{S_\mu, T_\mu, Y_{\mu\nu}\}$, which can be placed in the following lattice of algebras:

$$\mathrm{SO}(8) \supset \left\{ \begin{array}{c} \mathrm{SO}_S(5) \otimes \mathrm{SO}_T(3) \\ \mathrm{SO}(6) \\ \mathrm{SO}_T(5) \otimes \mathrm{SO}_S(3) \end{array} \right\} \supset \mathrm{SO}_S(3) \otimes \mathrm{SO}_T(3). \tag{66}$$

By use of the explicit form of the generators of SO(8) and its subalgebras, and their commutation relations [44], the following relations can be shown to hold:

$$\begin{aligned} S_+ \cdot S_- &= \frac{1}{2} C_2[\mathrm{SO}_T(5)] - \frac{1}{2} C_2[\mathrm{SO}_T(3)] - \frac{1}{8}(2\Omega - n)(2\Omega - n + 6), \\ S_+ \cdot S_- + P_+ \cdot P_- &= \frac{1}{2} C_2[\mathrm{SO}(8)] - \frac{1}{2} C_2[\mathrm{SO}(6)] - \frac{1}{8}(2\Omega - n)(2\Omega - n + 12), \\ P_+ \cdot P_- &= \frac{1}{2} C_2[\mathrm{SO}_S(5)] - \frac{1}{2} C_2[\mathrm{SO}_S(3)] - \frac{1}{8}(2\Omega - n)(2\Omega - n + 6), \end{aligned} \tag{67}$$

with $\Omega = \sum_\ell (2\ell + 1)$. This shows that the interaction (64) in the three cases (i) $g_0 = 0$, (ii) $g_0' = 0$ and (iii) $g_0 = g_0'$, can be written as a combination of Casimir operators of algebras belonging to a chain of *nested* algebras of the lattice (66). They are thus the dynamical symmetries of the SO(8) model.

The nature of 'SO(8) superfluidity' can be illustrated in the specific example of the ground state of even–even $N = Z$ nuclei. In the SO(6) limit of the SO(8) model the exact ground-state solution can be written as [45]

$$(S_+ \cdot S_+ - P_+ \cdot P_+)^{n/4} \, |\mathrm{o}\rangle. \tag{68}$$

This shows that the superfluid solution acquires a *quartet* structure in the sense that it reduces to a condensate of bosons each of which corresponds to four nucleons. Since the boson in (68) is a scalar in spin and isospin, it can be thought of as an α particle; its orbital character, however, might be different from that of an actual α particle. A quartet structure is also present in the two SO(5) limits of the SO(8) model, which yields a ground-state wave function of the type (68) with either the first or the second term suppressed. A reasonable *ansatz* for the $N = Z$ ground-state wave function of the SO(8) pairing interaction (64) with arbitrary strengths g_0 and g_0' is therefore

$$(\cos\theta \; S_+ \cdot S_+ - \sin\theta \; P_+ \cdot P_+)^{n/4} \, |\mathrm{o}\rangle, \tag{69}$$

where θ is a parameter that depends on the ratio g_0/g_0'. The condensate (69) of α-like particles provides an excellent approximation to the $N = Z$ ground state of the pairing hamiltonian (64) for any combination of g_0 and g_0' [45]. It should nevertheless be stressed that, in the presence of both neutrons and protons in the valence shell, the pairing hamiltonian (64) is *not* a good approximation to a realistic shell-model hamiltonian which contains an important quadrupole component.

These results can be generalized to the case of several non-degenerate shells. In fact, the Richardson equations (57) are valid for the quasi-spin symmetry SU(2) but they are known for any Lie algebra [46]. Closed solutions have been obtained for a system of neutron and protons with a pairing interaction of pure isovector character and of equal isovector and isoscalar strength, based on the SO(5) and the SO(6) quasi-spin algebras, respectively [47, 48].

5.4. Rotation models

In the early days of nuclear physics, nuclei with a rotational-like spectrum were interpreted either with the liquid-drop model of Bohr and Mottelson [49] or with a *deformed* single-particle shell model of Nilsson [50]. An understanding of rotational phenomena in terms of the spherical shell model, however, was lacking. Elliott's SU(3) model [7] provides such an understanding from a symmetry perspective. Since SU(3) is based on Wigner's supermultiplet model, first a discussion of the latter should be given.

Wigner's supermultiplet model [8] assumes nuclear forces to be invariant under rotations in *spin* as well as *isospin* space. This invariance is expressed by the following commutation relations:

$$[H,S_\mu] = [H,T_\mu] = [H,Y_{\mu\nu}] = 0, \tag{70}$$

where

$$S_\mu = \sum_{k=1}^{A} s_\mu(k), \qquad T_\mu = \sum_{k=1}^{A} t_\mu(k), \qquad Y_{\mu\nu} = \sum_{k=1}^{A} s_\mu(k) t_\nu(k), \tag{71}$$

are the spin, isospin and Gamow–Teller-like operators, in terms of $s_\mu(k)$ and $t_\mu(k)$, the spin and isospin components of nucleon k. The 15 operators (71) generate the Lie algebra SU(4). According to the discussion in Section 2, any hamiltonian satisfying the conditions (70) has SU(4) symmetry, and this in addition to symmetries associated with the conservation of total spin S and total isospin T.

To obtain a qualitative understanding of SU(4) symmetry, it is instructive to analyze the case of two nucleons in an oscillator shell. Total antisymmetry of the wave function requires that the spatial part is symmetric and the spin–isospin part antisymmetric or *vice versa*. Both cases correspond to a different symmetry under SU(4), the first being antisymmetric and the second symmetric. The symmetry under a given algebra can characterized by a Young diagram, and for two particles the antisymmetric configuration is denoted as $[1,1]$, while the symmetric one is written as $[2,0]$.

This argument can be generalized to an arbitrary number of nucleons and the result emerges that the SU(4) quantum numbers specify the way in which the overall antisymmetry is distributed over the spatial and spin–isospin parts of the wave function. More formally, the orbital/spin–isospin decomposition is equivalent to the algebraic reduction

$$\begin{array}{ccccc} \mathrm{U}(4\Omega) & \supset & \mathrm{U}(\Omega) & \otimes & \mathrm{U}(4) \\ \downarrow & & \downarrow & & \downarrow \\ [1^n] & & [f_1,f_2,f_3,f_4] & & [\bar{f}_1,\bar{f}_2,\bar{f}_3,\bar{f}_4] \end{array}, \tag{72}$$

where Ω denotes the orbital shell size (*i.e.*, $\Omega = 1,3,6,\ldots$ for the *s*, *p*, *sd*,... shells). The U(4) algebra consists of the SU(4) generators (71) supplemented with the particle-number operator n. The overall antisymmetry $[1^n]$ of the wave function requires conjugate symmetry under U(Ω) and U(4), which defines the relation between $[f_1,f_2,f_3,f_4]$ and $[\bar{f}_1,\bar{f}_2,\bar{f}_3,\bar{f}_4]$: they have *conjugate* Young diagrams [51]. As an example, the symmetry classification of one and two particles in the *sd* shell is summarized in Table 1. The table also gives the more commonly used SU(4) labels which are related to those of

TABLE 1. Classification of one particle and two particles in the *sd* shell.

n	$[f_1,f_2,f_3,f_4]$	L	$[\bar{f}_1,\bar{f}_2,\bar{f}_3,\bar{f}_4]$	(λ,μ,ν)	(S,T)
1	[1]	0,2	[1]	(1,0,0)	$(\frac{1}{2},\frac{1}{2})$
2	[2,0]	$0^2,2^2,4$	[1,1]	(0,1,0)	(0,1),(1,0)
	[1,1]	1,2,3	[2,0]	(2,0,0)	(0,0),(1,1)

U(4) through

$$\lambda = \bar{f}_1 - \bar{f}_2, \qquad \mu = \bar{f}_2 - \bar{f}_3, \qquad \nu = \bar{f}_3 - \bar{f}_4. \tag{73}$$

The physical relevance of Wigner's supermultiplet classification is connected with the short-range attractive nature of the residual interaction as a result of which states with spatial symmetry are favoured energetically. To see this point, consider an extreme form of a short-range interaction, namely a delta interaction. It has a vanishing matrix element in a spatially antisymmetric two-nucleon state since in that case the wave function has zero probability of having $\bar{r}_1 = \bar{r}_2$. In contrast, the matrix element is attractive in the spatially symmetric case with $[1,1]$ U(4) symmetry. Again, this result can be generalized to many nucleons, leading to the conclusion that the energy of a state depends on its SU(4) labels.

Wigner's supermultiplet model is a nuclear *LS*-coupling scheme. It is strongly broken by the large spin–orbit coupling in the nuclear mean field and, as a result, the SU(4) model is not considered as realistic any longer. In spite of its limited applicability, Wigner's idea remains important because it demonstrates the connection between the short-range character of the residual interaction and the spatial symmetry of the many-body wave function. The break down of SU(4) symmetry is a consequence of the spin–orbit term in the nuclear many-body hamiltonian (41) which does not satisfy the second and third commutator in (70). The spin–orbit term breaks SU(4) symmetry [SU(4) representations are admixed by it] and does so increasingly in heavier nuclei since the energy splitting of the spin doublets $\ell - \frac{1}{2}$ and $\ell + \frac{1}{2}$ increases with nucleon number A. In addition, SU(4) symmetry is also broken by the Coulomb interaction—an effect that also increases with A—and by spin-dependent residual interactions.

The break down of SU(4) symmetry with increasing nuclear mass number A can be illustrated with Gamow–Teller β decay [52] and with nuclear binding energies [53]. A simple way to represent the latter effect involves the double differences of nuclear binding energies [54, 55],

$$\delta V_{\rm np}(N,Z) = \frac{1}{4}[B(N,Z) - B(N-2,Z) - B(N,Z-2) + B(N-2,Z-2)], \tag{74}$$

where $B(N,Z)$ is the binding energy of a nucleus with N neutrons and Z protons and where N and Z are assumed even. The quantity $\delta V_{\rm np}(N,Z)$ acts as a filter to isolate the interaction between neutrons and protons. Particularly large values of $\delta V_{\rm np}(N,Z)$ are found for $N = Z$ [56]. The erosion of this $N = Z$ enhancement with mass number A provides a proof of the breaking of SU(4) symmetry. An example is shown in Fig. 8 which shows on the left the measured double binding energy $\delta V_{\rm np}(N,Z)$ for even–even

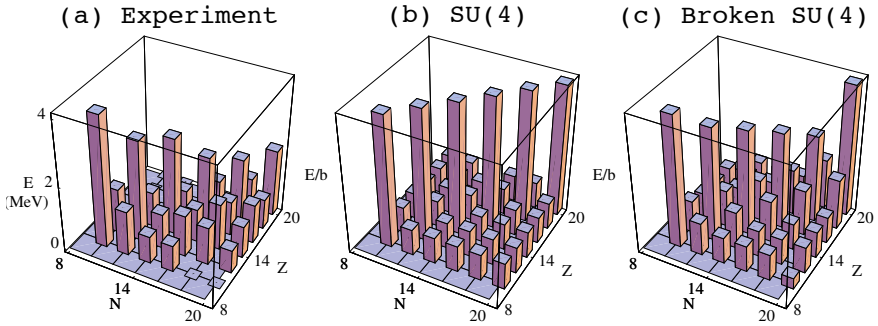

FIGURE 8. Barchart representation of double binding energy differences (a) as observed in even–even *sd*-shell nuclei [32], (b) as predicted by Wigner's unbroken SU(4) symmetry, and (c) as obtained by taking a mixture of first- and second-favoured SU(4) representations. The x and y coordinates of the centre of a cuboid define N and Z and its height z defines $\delta V_{\rm np}(N,Z)$. An empty square indicates that the data are lacking.

nuclei in the *sd* shell. The SU(4) result of Fig. 8b is obtained by assuming a nuclear binding energy of the form $a+b\langle C_2[\mathrm{SU}(4)]\rangle$ where a and b are coefficients depending smoothly on mass number and $\langle C_2[\mathrm{SU}(4)]\rangle$ is the eigenvalue of the quadratic Casimir of SU(4) in the favoured SU(4) representation [57]. As long as the departure from SU(4) symmetry is not too important, its breaking can be investigated by assuming a nuclear ground state which does not correspond entirely to the favoured SU(4) representation but contains an admixture of the next-favoured SU(4) representation. These admixtures will modify the behaviour of $\delta V_{\rm np}(N,Z)$ at $N\sim Z$. This is illustrated in Fig. 8c where $\delta V_{\rm np}(N,Z)$ is plotted by taking a varying mixture of first- and second-favoured SU(4) representations. As the mass of the nucleus increases, one notes indeed a decrease of the $N=Z$ enhancement effect for $\delta V_{\rm np}(N,Z)$, roughly consistent with the experimental observations. An exceptional point occurs for $N=Z=20$ where the calculation is unrealistic since ^{40}Ca is taken as doubly closed and hence corresponds to a unique SU(4) representation with no possible admixtures.

In Wigner's supermultiplet model the spatial part of the wave function is left unspecified. It is only assumed that the total orbital angular momentum L is a good quantum number. The main feature of Elliott's model [7] is that it provides an orbital classification which incorporates *rotational* characteristics. Elliott's model of rotation presupposes Wigner's SU(4) classification and assumes in addition that the residual interaction has a quadrupole character, a reasonable hypothesis if the valence shell contains neutrons and protons. With reference to the hamiltonian (41), one requires that it reduces to

$$H=\sum_{k=1}^{A}\left(\frac{p_k^2}{2m_{\rm n}}+\frac{1}{2}m_{\rm n}\omega^2 r_k^2\right)+V_{\rm quadrupole}, \tag{75}$$

where $V_{\rm quadrupole}=-g_2 Q\cdot Q$ contains a quadrupole operator

$$Q_\mu=\sqrt{\frac{3}{2}}\left[\sum_{k=1}^{A}\frac{1}{b^2}(\bar r_k\times\bar r_k)^{(2)}_\mu+\frac{b^2}{\hbar^2}\sum_{k=1}^{A}(\bar p_k\times\bar p_k)^{(2)}_\mu\right], \tag{76}$$

in terms of coordinates $\bar{r}_k$ and momenta $\bar{p}_k$, and where b is the oscillator length parameter (32). Note that $Q\cdot Q$ contains one-body ($k=l$) as well as two-body ($k\neq l$) terms.

To recognize that the shell-model hamiltonian (75) is analytically solvable, it is best to write it in second-quantized form. Because of its symmetric structure in $\bar{r}$ and $\bar{p}$, the quadrupole operator Q_μ does not couple to states outside a given valence shell and particle creation operators $a^\dagger_{\ell m_\ell s m_s t m_t}$ can be assigned the ℓ quantum number of that shell, together with spin and isospin labels. The quadrupole operator (76) can then be rewritten as (see Chapter 30 of Ref. [21])

$$Q_\mu = \sum_\ell \sqrt{8(2\ell+1)}\,(a^\dagger_{\ell st}\times\tilde{a}_{\ell st})^{(200)}_{\mu 00}, \tag{77}$$

where $\tilde{a}_{\ell m_\ell s m_s t m_t} = (-)^{\ell-m_\ell+s-m_s+t-m_t} a_{\ell -m_\ell s -m_s t -m_t}$. By construction, the quadrupole operator (76) is a scalar in spin and isospin, as it does not change either of them, and a tensor in orbital angular momentum. Likewise, the orbital angular momentum operator, $L_\mu = \sum_k (\bar{r}_k \wedge \bar{p}_k)_\mu/\hbar$, reads in second quantization

$$L_\mu = \sum_\ell \sqrt{\frac{4\ell(\ell+1)(2\ell+1)}{3}}\,(a^\dagger_{\ell st}\times\tilde{a}_{\ell st})^{(100)}_{\mu 00}. \tag{78}$$

The hamiltonian (75) can thus be rewritten as

$$H = \hbar\omega\left(N+\tfrac{3}{2}\right) - g_2 Q\cdot Q, \tag{79}$$

where N is an operator that counts the number of oscillator quanta. For a given number of nucleons in the valence shell the first term in (79) reduces to a constant; the second term, however, generates a spectrum as can be seen as follows.

The hamiltonian (75) satisfies the commutation relations (70) and hence has SU(4) symmetry. Its additional symmetry character depends on the orbital space available to the valence nucleons. With reference to the classification (72), the operators L_μ and Q_μ are scalar in spin and isospin and hence are generators of U(Ω). Furthermore, from their explicit expressions (77) and (78) one derives the commutation relations

$$\begin{aligned}
[Q_\mu, Q_\nu] &= 3\sqrt{10}\,\langle 2\mu\ 2\nu|1\mu+\nu\rangle L_{\mu+\nu},\\
[L_\mu, Q_\nu] &= -\sqrt{6}\,\langle 1\mu\ 2\nu|2\mu+\nu\rangle Q_{\mu+\nu},\\
[L_\mu, L_\nu] &= -\sqrt{2}\,\langle 1\mu\ 1\nu|1\mu+\nu\rangle L_{\mu+\nu},
\end{aligned} \tag{80}$$

which show that they generate an SU(3) Lie algebra that must then be a subalgebra of U(Ω). With the commutation relations (80) it can also be shown that the quadratic combination $Q\cdot Q + 3L\cdot L$ commutes with all generators of SU(3). The quadrupole interaction is thus a combination of Casimir operators,

$$Q\cdot Q = 4C_2[\mathrm{SU}(3)] - 3L\cdot L = 4C_2[\mathrm{SU}(3)] - 3C_2[\mathrm{SO}(3)], \tag{81}$$

and it follows that the hamiltonian (75) has the eigenvalues

$$E(\lambda,\mu,L) = E_0 - g_2\left[4(\lambda^2+\mu^2+\lambda\mu+3\lambda+3\mu) - 3L(L+1)\right], \tag{82}$$

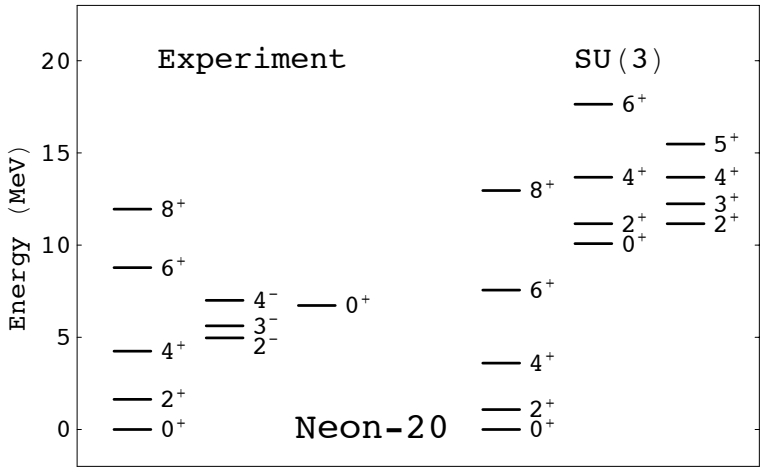

FIGURE 9. Observed [26] low-energy spectrum of ^{20}Ne (left) compared with the lowest SU(3) rotational bands (right). Levels are labelled by their angular momentum and parity J^π. The SU(3) spectrum is generated with a quadrupole interaction $-g_2 Q\cdot Q$ with strength $g_2 = 0.06$ MeV.

where E_0 is a constant energy associated with the first term in the hamiltonian (79). The quadrupole interaction implies the orbital reduction

$$\begin{array}{ccccc} \mathrm{U}(\Omega) & \supset & \mathrm{SU}(3) & \supset & \mathrm{SO}(3) \\ \downarrow & & \downarrow & & \downarrow \\ [f_1,f_2,f_3,f_4] & & (\lambda,\mu) & & K_L L \end{array} , \tag{83}$$

and represents an example of dynamical symmetry breaking. The degeneracy within a given Wigner supermultiplet is lifted (dynamically) by the quadrupole interaction.

A simple illustration of SU(3) dynamical symmetry is shown in Fig. 9. The nucleus ^{20}Ne contains two neutrons and two protons in the *sd* shell ($\Omega = 6$) above the ^{16}O closed-shell configuration. These four nucleons can acquire a spatially symmetric configuration, leading to $[f_1,f_2,f_3,f_4] = [4,0,0,0] \equiv [4]$ in U(6). All states in this symmetric configuration correspond to a single supermultiplet with labels $[\bar{f}_1,\bar{f}_2,\bar{f}_3,\bar{f}_4] = [1,1,1,1] \equiv [1^4]$ in U(4) and with $S = T = 0$. The degeneracy of this supermultiplet is lifted by the residual quadrupole interaction which gives rise to the SU(3) spectrum shown in Fig. 9. This interaction separates the different SU(3) multiplets (or representations) which can be $(\lambda,\mu) = (8,0)$, (4,2), (0,4) or (2,0). The allowed values of the total orbital angular momentum L (and, since $S = 0$, of the total angular momentum J) follow from the $\mathrm{SU}(3) \supset \mathrm{SO}(3)$ reduction rule [7]. For the lowest SU(3) multiplet with $(\lambda,\mu) = (8,0)$ they are $L = 0,2,4,6,8$. The observed angular momenta J of the states and their excitation energies as a function of J are approximately consistent with those of a rotational band with $K = 0$ projection of the total angular momentum on the axis of symmetry. The SU(3) model predicts this band to terminate at $J^\pi = 8^+$ which is consistent with the observations since the lowest $J^\pi = 10^+$ occurs at 27.5 MeV [26], well above the energy expected from a rotational behaviour. The experimental spectrum of ^{20}Ne contains many more levels than those in the $K^\pi = 0^+$ band, the lowest of which are shown in Fig. 9. States of four nucleons in the *sd* shell have positive parity and, consequently, the observed negative-parity levels necessarily must involve a (particle–hole) excitation

outside this shell. The first-excited 0^+ level possibly belongs to the next SU(3) multiplet with $(\lambda,\mu)=(4,2)$, also shown in Fig. 9, containing the levels $L=0,2^2,3,4^2,5,6$. Alternatively, it may correspond to a two-particle–two-hole excitation outside the *sd* shell.

The importance of Elliott's idea is that it gives rise to a rotational classification of states through mixing of spherical configurations. With the SU(3) model it was shown, for the first time, how deformed nuclear shapes may arise out of the spherical shell model. As a consequence, Elliott's work bridged the gap between the nuclear shell model and the liquid-drop model which up to that time (1958) existed as separate views of the nucleus.

At this point we can summarize the situation as follows. Elliott's SU(3) model provides a natural explanation of rotational phenomena, ubiquitous in nuclei, but it does so by assuming Wigner's SU(4) symmetry which is known to be badly broken in most nuclei. This puzzle has motivated much work since Elliott: How can rotational phenomena in nuclei be understood starting from a jj-coupling scheme induced by the spin–orbit term in the nuclear mean field? Arguably the most successful way to do so and to extend the applications of the SU(3) model to heavy nuclei is based upon the concept of pseudo-spin symmetry. The starting point for the explanation of this symmetry is the single-particle part of the hamiltonian (41),

$$H_{\rm ip}=\sum_{k=1}^{A}\left(\frac{p_k^2}{2m_{\rm n}}+\frac{1}{2}m_{\rm n}\omega^2 r_k^2+\zeta_{\rm oo}\bar{\ell}_k\cdot\bar{\ell}_k+\zeta_{\rm so}\bar{\ell}_k\cdot\bar{s}_k\right). \tag{84}$$

For $\zeta_{\rm oo}=\zeta_{\rm so}=0$ a three-dimensional isotropic harmonic oscillator is obtained which exhibits degeneracies associated with U(3) symmetry. For arbitrary non-zero values of $\zeta_{\rm oo}$ and $\zeta_{\rm so}$ this symmetry is broken. However, for the particular combination $4\zeta_{\rm oo}=\zeta_{\rm so}$ some degree of degeneracy, associated with a so-called pseudo-spin symmetry, is restored in the spectrum of $H_{\rm ip}$.

To understand the nature of pseudo-spin symmetry, consider the unitary transformation

$$U=\sum_{k=1}^{A}u_k,\qquad u_k=2i\frac{\bar{s}_k\cdot\bar{r}_k}{r_k}, \tag{85}$$

and apply this transformation to the hamiltonian (84). One finds

$$U^{-1}H_{\rm ip}U=\sum_{k=1}^{A}\left(\frac{p_k^2}{2m_{\rm n}}+\frac{1}{2}m_{\rm n}\omega^2 r_k^2+\zeta_{\rm oo}\bar{\ell}_k\cdot\ell_k+(4\zeta_{\rm oo}-\zeta_{\rm so})\bar{\ell}_k\cdot\bar{s}_k\right)+C, \tag{86}$$

where $C=A(\hbar\omega+2\zeta_{\rm oo}-\zeta_{\rm so})$ is a constant. The original and transformed hamiltonians have the same eigenspectrum since they are related, up to the constant, by a unitary transformation. This shows that for $4\zeta_{\rm oo}=\zeta_{\rm so}$ the spectrum of $H_{\rm ip}$ is identical (up to a constant) to that of a single-particle hamiltonian with only an orbit–orbit and no spin–orbit term. This results in single-particle orbits with $j=\ell+\frac{1}{2}$ and $j=(\ell+2)-\frac{1}{2}$ being degenerate for *all* values of ℓ. These single-particle orbits can be considered as originating from a pseudo-orbital angular momentum $\tilde{\ell}=\ell+1$, in the presence of zero pseudo-spin–orbit splitting $\tilde{\ell}\cdot\tilde{s}$.

Pseudo-spin symmetry has a long history in nuclear physics. The existence of nearly degenerate pseudo-spin doublets in the nuclear mean-field potential was pointed out more than forty years ago by Hecht and Adler [58] and by Arima *et al.* [59] who noted that, because of the small pseudo-spin–orbit splitting, pseudo-LS (or $\tilde{L}\tilde{S}$) coupling should be a reasonable starting point in medium-mass and heavy nuclei where LS coupling becomes unacceptable. With $\tilde{L}\tilde{S}$ coupling as a premise, an pseudo-SU(3) model can be constructed [60] in much the same way as Elliott's SU(3) model can be defined in LS coupling. The formal definition of the pseudo-spin transformation (85) in terms of a helicity operator was given in Refs. *et al.* [61, 62]. Finally, it is only many years after its original suggestion that Ginocchio showed pseudo-spin to be a symmetry of the Dirac equation which occurs if the scalar and vector potentials are equal in size but opposite in sign [63].

The models discussed so far all share the property of being confined to a single shell, either an oscillator or a pseudo-oscillator shell. A full description of nuclear collective motion requires correlations that involve configurations outside a single shell. The proper framework for such correlations invokes the concept of a non-compact algebra which, in contrast to a compact one, can have infinite-dimensional unitary representations. The latter condition is necessary since the excitations into higher shells can be infinite in number. The inclusion of excitations into higher shells of the harmonic oscillator, was achieved by Rosensteel and Rowe by embedding the SU(3) algebra into the symplectic algebra Sp(3,R) [64].

6. SUMMARY

In these lecture notes an introduction was given to the notions of symmetry and dynamical symmetry (or spectrum generating algebra). Their use in the solution of the nuclear many-body problem was described. As an example of these techniques, the U(3) symmetry of the harmonic oscillator in three dimensions was discussed in detail. A review was given of the shell model, with particular emphasis on the application of group-theoretical techniques in the context of this model.

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